Vypočítejte první a druhé parciální derivace funkcí:

1. $f(x,y) = 3(x)^2 - 5(y)^3 + 1$

$$\frac{\partial f(x,y)}{\partial x} = 6x \qquad \frac{\partial f(x,y)}{\partial y} = -15(y)^2$$

$$\frac{\partial^2 f(x,y)}{\partial x^2} = 6 \qquad \frac{\partial^2 f(x,y)}{\partial y^2} = -30y \qquad \frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial y \partial x} = 0$$

2. $f(x,y) = 4\sqrt[3]{x^5} - \ln y^2 = 4x^{\frac{5}{3}} - 2\ln y$

$$\frac{\partial f(x,y)}{\partial x} = \frac{20}{3}x^{\frac{2}{3}} \qquad \frac{\partial f(x,y)}{\partial y} = -\frac{2}{y}$$

$$\frac{\partial^2 f(x,y)}{\partial x^2} = \frac{40}{9}x^{-\frac{1}{3}} \qquad \frac{\partial^2 f(x,y)}{\partial y^2} = \frac{2}{y^2} \qquad \frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial y \partial x} = 0$$

3. $f(x,y) = 5x^3y^2 - x^2y^3$

$$\frac{\partial f(x,y)}{\partial x} = 15x^2y^2 - 2xy^3 \qquad \frac{\partial f(x,y)}{\partial y} = 10x^3y - 3x^2y^2$$

$$\frac{\partial^2 f(x,y)}{\partial x^2} = 30xy^2 - 2y^3 \qquad \frac{\partial^2 f(x,y)}{\partial y^2} = 10x^3 - 6x^2y$$

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial y \partial x} = 30x^2y - 6xy^2$$

4. $f(x,y) = x^y$

$$\frac{\partial f(x,y)}{\partial x} = yx^{y-1} \qquad \frac{\partial f(x,y)}{\partial y} = x^y \ln x$$

$$\frac{\partial^2 f(x,y)}{\partial x^2} = y(y-1)x^{y-2} \qquad \frac{\partial^2 f(x,y)}{\partial y^2} = \ln x\, x^y\, \ln x = x^y \ln^2 x$$

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial y \partial x} = yx^{y-1}\ln x + x^{y-1}$$

5. $f(x,y) = x\sin(x+y)$

$$\frac{\partial f(x,y)}{\partial x} = \sin(x+y) + x\cos(x+y) \qquad \frac{\partial f(x,y)}{\partial y} = x\cos(x+y)$$

$$\frac{\partial^2 f(x,y)}{\partial x^2} = \cos(x+y) + \cos(x+y) + x\sin(x+y) = 2\cos(x+y) - x\sin(x+y)$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = -x\sin(x+y)$$

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial y \partial x} = \cos(x+y) - x\sin(x+y)$$

Vypočítejte první parciální derivace funkcí:

6. $f(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$

$$\frac{\partial f(x,y)}{\partial x} = \frac{3x^2(x^2+y^2)-(x^3+y^3)2x}{(x^2+y^2)^2} = \frac{3x^4+3x^2y^2-2x^4-2xy^3}{(x^2+y^2)^2} = \frac{x^4+3x^2y^2-2xy^3}{(x^2+y^2)^2}$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{3y^2(x^2+y^2)-(x^3+y^3)2y}{(x^2+y^2)^2} = \frac{3x^2y^2+3y^4-2x^3y-2y^4}{(x^2+y^2)^2} = \frac{y^4+3x^2y^2-2x^3y}{(x^2+y^2)^2}$$

7. $f(x,y) = (\sin x)^{\cos y}$

$$\frac{\partial f(x,y)}{\partial x} = \cos y (\sin x)^{cosy-1}\cos x \qquad \frac{\partial f(x,y)}{\partial y} = -\sin y(\sin x)^{\cos y}\ln(\sin x)$$

8. $f(x,y) = \tan\left(\dfrac{x^2}{y}\right)$

$$\frac{\partial f(x,y)}{\partial x} = \frac{1}{\cos^2\left(\frac{x^2}{y}\right)}\frac{2x}{y} \qquad \frac{\partial f(x,y)}{\partial y} = \frac{1}{\cos^2\left(\frac{x^2}{y}\right)}\frac{-x^2}{y^2}$$

9. $f(x,y) = \left(\dfrac{1}{3}\right)^{\frac{y}{x}}$

$$\frac{\partial f(x,y)}{\partial x} = \left(\frac{1}{3}\right)^{\frac{y}{x}}\ln 3\frac{y}{x^2} \qquad \frac{\partial f(x,y)}{\partial y} = -\left(\frac{1}{3}\right)^{\frac{y}{x}}\ln 3\frac{1}{x}$$

10. $f(x,y) = xe^{\frac{y}{x}}$

$$\frac{\partial f(x,y)}{\partial x} = e^{\frac{y}{x}} - xe^{\frac{y}{x}}\frac{y}{x^2} \qquad \frac{\partial f(x,y)}{\partial y} = xe^{\frac{y}{x}}\frac{1}{x} = e^{\frac{y}{x}}$$

11. $f(x,y) = \ln\dfrac{x-y}{x+y}$

$$\frac{\partial f(x,y)}{\partial x} = \frac{1}{\frac{x-y}{x+y}}\frac{(x+y)-(x-y)}{(x+y)^2} = \frac{2y}{(x^2-y^2)}$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{1}{\frac{x-y}{x+y}}\frac{-1(x+y)-(x-y)}{(x+y)^2} = \frac{-2x}{(x^2-y^2)}$$

12. $f(x,y) = \ln(x+\sqrt{x^2+y^2})$

$$\frac{\partial f(x,y)}{\partial x} = \frac{1}{x+\sqrt{x^2+y^2}}\left(1+\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}}\right) = \frac{1}{x+\sqrt{x^2+y^2}}\left(\frac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2}}\right) = \frac{1}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{1}{x+\sqrt{x^2+y^2}}\frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}} = \frac{y}{x\sqrt{x^2+y^2}+x^2+y^2}$$

13. $f(x,y) = \arcsin\dfrac{x}{\sqrt{x^2+y^2}}$

$$\frac{\partial f(x,y)}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2}}\frac{\sqrt{x^2+y^2}-x\frac{1}{2}\frac{1}{\sqrt{x^2+y^2}}2x}{x^2+y^2} =$$

$$= \frac{1}{\sqrt{\frac{x^2+y^2-x^2}{x^2+y^2}}}\frac{x^2+y^2-x^2}{(x^2+y^2)\sqrt{x^2+y^2}} = \frac{y}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2}}x\frac{-1}{2}\frac{1}{\sqrt{(x^2+y^2)^3}}2y =$$

$$= \frac{1}{\sqrt{\frac{x^2+y^2-x^2}{x^2+y^2}}}\frac{-xy}{(x^2+y^2)\sqrt{x^2+y^2}} = \frac{-x}{x^2+y^2}$$

14. $f(x,y) = \arctan\dfrac{x-y}{x+y}$

$$\frac{\partial f(x,y)}{\partial x} = \frac{1}{1+\left(\frac{x-y}{x+y}\right)^2}\frac{(x+y)-(x-y)}{(x+y)^2} =$$

$$= \frac{(x+y)^2}{x^2+2yx+y^2+x^2-2yx+y^2}\frac{x+y-x+y}{(x+y)^2} = \frac{y}{x^2+y^2}$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{1}{1+\left(\frac{x-y}{x+y}\right)^2}\frac{-1(x+y)-(x-y)}{(x+y)^2} =$$

$$= \frac{(x+y)^2}{x^2+2yx+y^2+x^2-2yx+y^2}\frac{-x-y-x+y}{(x+y)^2} = \frac{-x}{x^2+y^2}$$

15. $f(x,y) = (x+y)^y$

Využijeme vzorce: $A^B = e^{B\ln A}$

$$f(x,y) = e^{y\ln(x+y)}$$

$$\frac{\partial f(x,y)}{\partial x} = e^{y\ln(x+y)}y\frac{1}{x+y} \qquad \frac{\partial f(x,y)}{\partial y} = e^{y\ln(x+y)}\left[\ln(x+y)+y\frac{1}{x+y}\right]$$

16. $f(x,y) = (x)^{x^y}$

Využijeme vzorce: $A^B = e^{B\ln A}$

$$f(x,y) = e^{x^y\ln x}$$

$$\frac{\partial f(x,y)}{\partial x} = e^{x^y\ln x}\left(yx^{y-1}\ln x + x^y\frac{1}{x}\right) \qquad \frac{\partial f(x,y)}{\partial y} = e^{x^y\ln x}(\ln x)^2x^y$$

17. $f(x,y) = xy \cdot e^{\sin(\pi xy)}$

$$\frac{\partial f(x,y)}{\partial x} = y \cdot e^{\sin(\pi xy)}\,[1 + \pi xy \cos(\pi xy)]$$

$$\frac{\partial f(x,y)}{\partial y} = x \cdot e^{\sin(\pi xy)}\,[1 + \pi xy \cos(\pi xy)]$$

18. $f(x,y) = \ln \frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}$

$$\frac{\partial f(x,y)}{\partial x} = \frac{1}{\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}} \frac{\left(\frac{1}{2}\frac{1}{\sqrt{x^2+y^2}}2x - 1\right)(\sqrt{x^2+y^2}+x) - (\sqrt{x^2+y^2}-x)\left(\frac{1}{2}\frac{1}{\sqrt{x^2+y^2}}2x + 1\right)}{(\sqrt{x^2+y^2}+x)^2} =$$

$$= \frac{\left(x - \sqrt{x^2+y^2}\right)\left(x + \sqrt{x^2+y^2}\right) - (\sqrt{x^2+y^2}-x)(\sqrt{x^2+y^2}+x)}{(\sqrt{x^2+y^2}+x)(\sqrt{x^2+y^2}-x)\sqrt{x^2+y^2}} =$$

$$= \frac{\left(x - \sqrt{x^2+y^2}\right)\left(x + \sqrt{x^2+y^2}\right) + (x - \sqrt{x^2+y^2})(x + \sqrt{x^2+y^2})}{y^2\sqrt{x^2+y^2}} =$$

$$= \frac{[x^2 - (x^2+y^2)] + [x^2 - (x^2+y^2)]}{y^2\sqrt{x^2+y^2}} = \frac{-2}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2}-x} \frac{\left(\frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}}\right)\left[(\sqrt{x^2+y^2}+x) - (\sqrt{x^2+y^2}-x)\right]}{(\sqrt{x^2+y^2}+x)^2} =$$

$$= \frac{2xy}{(\sqrt{x^2+y^2}-x)(\sqrt{x^2+y^2}+x)} = \frac{2x}{y\sqrt{x^2+y^2}}$$

Vypočítejte parciální derivace tří proměných:

19. $$f(x,y,z)=\sqrt{x^2+y^2+z^2}$$

$$\frac{\partial f(x,y,z)}{\partial x}=\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}}=\frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial f(x,y,z)}{\partial y}=\frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}}=\frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial f(x,y,z)}{\partial z}=\frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}}=\frac{z}{\sqrt{x^2+y^2+z^2}}$$

20. $$f(x,y,z)=(xy)^z$$

Využijeme vzorce: $A^B=e^{B\ln A}$

$$f(x,y,z)=e^{z\ln(xy)}$$

$$\frac{\partial f(x,y,z)}{\partial x}=e^{z\ln(xy)}z\frac{1}{xy}y=e^{z\ln(xy)}\frac{z}{x}=(xy)^z\frac{z}{x}$$

$$\frac{\partial f(x,y,z)}{\partial y}=e^{z\ln(xy)}z\frac{1}{xy}x=e^{z\ln(xy)}\frac{z}{y}=(xy)^z\frac{z}{y}$$

$$\frac{\partial f(x,y,z)}{\partial z}=e^{z\ln(xy)}\ln(xy)=(xy)^z\ln(xy)$$