1. Nechť u,resp. $\vec{f}$ označuje skalární, resp. vektorovou funkci třídy C^1. Použitím rozkladu (dále naznačeného dolními vodorovnými svorkami) podle pravidla

$$\mathrm{div}(u\vec{f}) = u\mathrm{div}\vec{f} + \vec{f} \cdot \bigtriangledown u$$

vypočítejte divergenci pole elektricke indukce (E je konstantní veličina)

$$\vec{D} = \frac{E}{\parallel \vec{r} \parallel^3}\vec{r} = \frac{E}{\sqrt{(x^2+y^2+z^2)^3}}(x\vec{i} + y\vec{j} + z\vec{k})$$

Vektor elektrické idukce můžeme rozdělit na skalární část u a vektorovou část $\vec{f}$.

$$u = \frac{E}{\sqrt{(x^2+y^2+z^2)^3}} \qquad a \qquad \vec{f} = (x\vec{i} + y\vec{j} + z\vec{k})$$

Do zadaného vztahu potřebujeme div$\vec{f}$(divergenci vektorové funcke f) a $\bigtriangledown u$ (gradient skalární funkce u).

Dle definice $\vec{f} = P\vec{i} + Q\vec{j} + R\vec{k}$ je divergence rovna:

$$\mathrm{div}\vec{f} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

a gradient skalární funkce u:

$$\bigtriangledown u = \frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j} + \frac{\partial u}{\partial z}\vec{k}.$$

Tedy pro naše zadání je div$\vec{f}$ a $\bigtriangledown u$:

$$\mathrm{div}\vec{f} = (1+1+1)$$

$$\frac{\partial u}{\partial x} = \frac{-3E}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}2x \qquad \frac{\partial u}{\partial y} = \frac{-3E}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}2y \qquad \frac{\partial u}{\partial z} = \frac{-3E}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}2z$$

$$\bigtriangledown u = -3E(x^2+y^2+z^2)^{-\frac{5}{2}}x\vec{i} + -3E(x^2+y^2+z^2)^{-\frac{5}{2}}y\vec{j} + -3E(x^2+y^2+z^2)^{-\frac{5}{2}}z\vec{k}$$

$$\bigtriangledown u = -3E\frac{1}{\sqrt{(x^2+y^2+z^2)^5}}(x\vec{i} + y\vec{j} + z\vec{k})$$

Doplníme do předpisu:

$$\mathrm{div}(u\vec{f}) = \frac{E}{\sqrt{(x^2+y^2+z^2)^3}} \cdot 3 + (x\vec{i} + y\vec{j} + z\vec{k}) \cdot \frac{-3E}{\sqrt{(x^2+y^2+z^2)^5}}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\mathrm{div}(u\vec{f}) = \frac{3E}{\sqrt{(x^2+y^2+z^2)^3}} + \frac{-3E}{\sqrt{(x^2+y^2+z^2)^5}}(x\vec{i} + y\vec{j} + z\vec{k})^2$$

$$\mathrm{div}(u\vec{f}) = \frac{3E}{\sqrt{(x^2+y^2+z^2)^3}} - \frac{3E}{(x^2+y^2+z^2)\sqrt{(x^2+y^2+z^2)^3}}(x\vec{i}+y\vec{j}+z\vec{k})^2$$

POZN:
$\vec{i}\cdot\vec{i} = (1,0,0)(1,0,0) = 1+0+0 = 1$
$\vec{j}\cdot\vec{j} = (0,1,0)(0,1,0) = 0+1+0 = 1$
$\vec{i}\cdot\vec{j} = (1,0,0)(0,1,0) = 0+0+0 = 0$
$\vec{j}\cdot\vec{k} = (0,1,0)(0,0,1) = 0+0+0 = 0$ atd.

$$\mathrm{div}(u\vec{f}) = \frac{3E}{\sqrt{(x^2+y^2+z^2)^3}} - \frac{3E}{(x^2+y^2+z^2)\sqrt{(x^2+y^2+z^2)^3}}(x^2+y^2+z^2)$$

$$\mathrm{div}(u\vec{f}) = \frac{3E}{\sqrt{(x^2+y^2+z^2)^3}} - \frac{3E}{\sqrt{(x^2+y^2+z^2)^3}} = 0$$

Divergence pole je rovna nule, jedaná se o pole neřídlové.

2. Operátorovým počtem dokažte, že pro funkce třídy C^2 platí $\mathrm{rotgrad}u = \vec{o}$ a $\mathrm{divrot}\vec{f} = 0$.

$$\mathrm{rotgrad}u = \mathrm{rot}\left(\frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j} + \frac{\partial u}{\partial z}\vec{k}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{vmatrix} =$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right)\right]\vec{i} + \left[\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right)\right]\vec{j} + \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\right]\vec{k} =$$

$$= \left[\frac{\partial^2 u}{\partial z\partial y} - \frac{\partial^2 u}{\partial y\partial z}\right]\vec{i} + \left[\frac{\partial^2 u}{\partial x\partial z} - \frac{\partial^2 u}{\partial z\partial x}\right]\vec{j} + \left[\frac{\partial^2 u}{\partial y\partial x} - \frac{\partial^2 u}{\partial x\partial y}\right]\vec{k} = 0\vec{i}+0\vec{j}+0\vec{k} = \vec{o}$$

$$\mathrm{divrot}\vec{f} = \mathrm{div}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \mathrm{div}\left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}\right] =$$

$$= \left[\frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] =$$

$$= \left(\frac{\partial^2 R}{\partial y\partial x} - \frac{\partial^2 Q}{\partial z\partial x} + \frac{\partial^2 P}{\partial z\partial y} - \frac{\partial^2 R}{\partial x\partial y} + \frac{\partial^2 Q}{\partial x\partial z} - \frac{\partial^2 P}{\partial y\partial z}\right) = 0$$

3.

$$\vec{f} = \left(\frac{x}{x^2+y^2+1}; \frac{y}{x^2+y^2+1}; 3z\right)$$

Jestliže mluvíme o poli konzervativním, pak je jeho rot$\vec{f} = 0$. Dle definice jsou funkce $P = \frac{x}{x^2+y^2+1}$, $Q = \frac{y}{x^2+y^2+1}$,$R = 3z$.

$$\mathrm{rot}\vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}\right]$$

$$\frac{\partial R}{\partial y} = 0 \quad \frac{\partial Q}{\partial z} = 0 \quad \frac{\partial P}{\partial z} = 0 \frac{\partial R}{\partial x} = 0 \quad \frac{\partial Q}{\partial x} = \frac{-2xy}{(x^2+y^2+1)^2} \quad \frac{\partial P}{\partial y} = \frac{-2xy}{(x^2+y^2+1)^2}$$

po dosazení do zadání obdržíme vztah:

$$= \left[(0-0)\,\vec{i} + (0-0)\,\vec{j} + \left(\frac{-2xy}{(x^2+y^2+1)^2} + \frac{2xy}{(x^2+y^2+1)^2}\right)\vec{k}\right] = 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{o}$$

4.