Najděte lokální extrémy funkce f

1.

$$f(x,y) = x^3 - 3xy + y^2 + y - 7$$

$f_x = 3x^2 - 3y, \quad f_y = -3x + 2y + 1$
Vyřešíme soustavu dvou rovnic o dvou neznámých a najdeme tak stacionární body
$3x^2 - 3y = 0 \Rightarrow y = x^2$
$-3x + 2y + 1 = 0 \Rightarrow -3x + 2x^2 + 1 = 0 \Rightarrow x_{1,2} = \frac{3 \pm \sqrt{1}}{4}$
$x_1 = 1 \Rightarrow y_1 = 1, \quad x_2 = \frac{1}{2} \Rightarrow y_2 = \frac{1}{4}$
Máme dva stacionární body $P_1 = [1,1], P_2 = \left[\frac{1}{2}, \frac{1}{4}\right]$.
Najdeme Hessovu matici.
$f_{xx} = 6x, \quad f_{xy} = -3, \quad f_{yy} = 2$

$$H(x,y) = \begin{pmatrix} 6x & -3 \\ -3 & 2 \end{pmatrix}$$

$\det H = 12x - 9$
$\det H(P_1) = 3 > 0 \Rightarrow$ extrém nastane, $\quad f_{xx}(P_1) = 6 > 0 \Rightarrow$ lokální minimum
$\det H(P_2) = -3 < 0 \Rightarrow$ extrém nenastane

2.

$$f(x,y) = 6xy - x^3 - y^3$$

$f_x = 6y - 3x^2, \quad f_y = 6x - 3y^2$
$6y - 3x^2 = 0 \Rightarrow y = \frac{x^2}{2}$
$6x - 3y^2 = 0 \Rightarrow 6x - \frac{3x^4}{4} = 0 \Rightarrow 3x(8 - x^3) = 0$
$x_1 = 0 \Rightarrow y_1 = 0, \quad x_2 = 2 \Rightarrow y_2 = 2$
$P_1 = [0,0], P_2 = [2,2]$
$f_{xx} = -6x, \quad f_{xy} = 6, \quad f_{yy} = -6y$

$$H(x,y) = \begin{pmatrix} -6x & 6 \\ 6 & -6y \end{pmatrix}$$

$\det H(x,y) = 36xy - 36$
$\det H(P_1) = -36 < 0 \Rightarrow$ extrém nenastane
$\det H(P_2) = 36 \cdot 4 - 36 > 0 \Rightarrow$ extrém nastane, $\quad f_{xx}(P_2) = -12 < 0 \Rightarrow$ lokální maximum

3.

$$f(x,y) = (x + y^2)\,\mathrm{e}^{\frac{x}{2}}$$

$f_x = \mathrm{e}^{\frac{x}{2}} + \frac{1}{2}(x + y^2)\,\mathrm{e}^{\frac{x}{2}}, \quad f_y = 2y\,\mathrm{e}^{\frac{x}{2}}$
$2y\,\mathrm{e}^{\frac{x}{2}} = 0 \Rightarrow y = 0$
$\mathrm{e}^{\frac{x}{2}} + \frac{1}{2}(x + y^2)\,\mathrm{e}^{\frac{x}{2}} = 0 \Rightarrow \mathrm{e}^{\frac{x}{2}}(1 + \frac{x}{2}) = 0 \Rightarrow x = -2$
$P = [-2, 0]$
$f_{xx} = \frac{1}{2}\,\mathrm{e}^{\frac{x}{2}} + \frac{1}{2}\,\mathrm{e}^{\frac{x}{2}} + \frac{1}{4}(x + y^2)\,\mathrm{e}^{\frac{x}{2}} = \mathrm{e}^{\frac{x}{2}} + \frac{1}{4}(x + y^2)\,\mathrm{e}^{\frac{x}{2}}$
$f_{xy} = y\,\mathrm{e}^{\frac{x}{2}}, \quad f_{yy} = 2\,\mathrm{e}^{\frac{x}{2}}$

$$H(x,y) = \begin{pmatrix} \mathrm{e}^{\frac{x}{2}} + \frac{1}{4}(x + y^2)\,\mathrm{e}^{\frac{x}{2}} & y\,\mathrm{e}^{\frac{x}{2}} \\ y\,\mathrm{e}^{\frac{x}{2}} & 2\,\mathrm{e}^{\frac{x}{2}} \end{pmatrix}$$

$\det H(x,y) = (\mathrm{e}^{\frac{x}{2}} + \frac{1}{4}(x + y^2)\,\mathrm{e}^{\frac{x}{2}})2\,\mathrm{e}^{\frac{x}{2}} - y^2\,\mathrm{e}^{x}$
$\det H(-2,0) = 2\,\mathrm{e}^{-1} + \frac{1}{2}\,\mathrm{e}^{-1} > 0 \Rightarrow$ extrém nastane, $\quad f_{xx}(-2,0) > 0 \Rightarrow$ lokální minimum

4.

$$f(x,y) = x\sqrt{y} - x^2 - y + 6x + 3$$

$f_x = \sqrt{y} - 2x + 6, \quad f_y = \frac{x}{2\sqrt{y}} - 1$
$\frac{x}{2\sqrt{y}} - 1 = 0 \Rightarrow \sqrt{y} = \frac{x}{2}$
$\sqrt{y} - 2x + 6 = 0 \Rightarrow \frac{x}{2} - 2x + 6 = 0 \Rightarrow x = 4$

$x = 4 \Rightarrow y = 4, \quad P = [4, 4]$
$f_{xx} = -2, \quad f_{xy} = \frac{1}{2\sqrt{y}}, \quad f_{yy} = -\frac{x}{4\sqrt{y^3}}$

$$H(x, y) = \begin{pmatrix} -2 & \frac{1}{2\sqrt{y}} \\ \frac{1}{2\sqrt{y}} & -\frac{x}{4\sqrt{y^3}} \end{pmatrix}$$

$\det H = \frac{x}{2\sqrt{y^3}} - \frac{1}{4y}$
$\det H(4, 4) = 1 - \frac{1}{16} > 0 \Rightarrow$ extrém nastane, $\quad f_{xx}(4, 4) = -2 \Rightarrow$ lokální maximum