1. $I = \iint\limits_A (x^2 + 2y^2 - 2x - 4y + 4)\, dx\, dy, \quad A:$ čtverec $< 0,2 > \times < 0,2 >$

Množina A je dána nerovnostmi $\begin{matrix} 0 & \leq & x & \leq & 2 \\ 0 & \leq & y & \leq & 2 \end{matrix}$

$$I = \int_0^2 \left(\int_0^2 (x^2 + 2y^2 - 2x - 4y + 4)\, dy \right) dx = \int_0^2 \left[x^2 y + 2\frac{y^3}{3} - 2xy - 4\frac{y^2}{2} + 4y \right]_{y=0}^{2} dx$$

$$= \int_0^2 (2x^2 + \frac{16}{3} - 4x - 8 + 8)\, dx = \left[2\frac{x^3}{3} + \frac{16}{3}x - 4\frac{x^2}{2} \right]_0^2 = \frac{16}{3} + \frac{32}{3} - \frac{16}{2} = \frac{48}{3} - 8 = 8$$

2. $I = \iint\limits_A x\, dx\, dy, \quad A:$ trojúhelník zadaný body $[0,0], [1,1], [0,1]$

Množina A je dána nerovnostmi $\begin{matrix} 0 & \leq & x & \leq & 1 \\ x & \leq & y & \leq & 1 \end{matrix}$

$$I = \int_0^1 \left(\int_x^1 x\, dy \right) dx = \int_0^1 [xy]_{y=x}^{y=1}\, dx = \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

3. $I = \iint\limits_A xy^2\, dx\, dy, \quad A:$ množina omezená křivkami $y^2 = x, x = 1$

Množina A je dána nerovnostmi $\begin{matrix} -1 & \leq & y & \leq & 1 \\ y^2 & \leq & x & \leq & 1 \end{matrix}$

$$I = \int_{-1}^1 \left(\int_{y^2}^1 xy^2\, dx \right) dy = \int_{-1}^1 \left[\frac{x^2}{2} y^2 \right]_{x=y^2}^{x=1} dy = \int_{-1}^1 \left(\frac{1}{2}y^2 - \frac{y^6}{2} \right) dy = \left[\frac{y^3}{6} - \frac{y^7}{14} \right]_{-1}^1$$

$$= \frac{1}{6} - \frac{1}{14} + \frac{1}{6} - \frac{1}{14} = \frac{1}{3} - \frac{1}{7} = \frac{4}{21}$$

4. $I = \iint\limits_A (x^2 + y^2)\, dx\, dy, \quad A:$ množina omezená křivkami $y = x, y = x + 2, y = 2, y = 6$

Množina A je dána nerovnostmi $\begin{matrix} 2 & \leq & y & \leq & 6 \\ y - 2 & \leq & x & \leq & y \end{matrix}$

$$I = \int_2^6 \left(\int_{y-2}^y (x^2 + y^2)\, dx \right) dy = \int_2^6 \left[\frac{x^3}{3} + y^2 x \right]_{x=y-2}^{x=y} dy = \int_2^6 \left(-\frac{(y-2)^3}{3} - y^2(y-2) + \frac{y^3}{3} + y^3 \right) dy$$

$$= -\int_2^6 \left(\frac{y^3 - 6y^2 + 12y - 8}{3} + y^3 - 2y^2 - \frac{y^3}{3} - y^3 \right) dy = -\int_2^6 \left(-4y^2 + 4y - \frac{8}{3} \right) dy = -\left[-\frac{4y^3}{3} + 4\frac{y^2}{2} - \frac{8}{3}y \right]_2^6$$

$$= \frac{4 \cdot 6^3}{3} - 2 \cdot 6^2 + \frac{8}{3} \cdot 6 + \left(\frac{-4 \cdot 8}{3} + 8 - \frac{16}{3} \right) = \frac{4 \cdot 6^3 - 2 \cdot 6^2 \cdot 3 + 8 \cdot 6 - 32 + 24 - 16}{3} = \frac{672}{3} = 224$$

5. $I = \iint\limits_A dx\, dy, \quad A:$ množina omezená křivkami $y = 6 - x^2, x + y - 4 = 0$

Množina A je dána nerovnostmi $\begin{matrix} -1 & \leq & x & \leq & 2 \\ 4 - x & \leq & y & \leq & 6 - x^2 \end{matrix}$

$$I = \int_{-1}^2 \left(\int_{4-x}^{6-x^2} dy \right) dx = \int_{-1}^2 (6 - x^2 - 4 + x) = \left[2x - \frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^2 = 4 - \frac{8}{3} + 2 - \left(-2 + \frac{1}{3} + \frac{1}{2} \right) = \frac{9}{2}$$

6. $I = \iint\limits_A dx\, dy, \quad A:$ množina omezená křivkami $y = x^2, y = 8 - x^2$

Množina A je dána nerovnostmi $\begin{matrix} -2 & \leq & x & \leq & 2 \\ x^2 & \leq & y & \leq & 8 - x^2 \end{matrix}$

$$I = \int_{-2}^2 \int_{x^2}^{8-x^2} dy\, dx = \int_{-2}^2 (8 - x^2 - x^2)\, dx = \left[8x - \frac{2x^3}{3} \right]_{-2}^2 = \left(16 - \frac{16}{3} \right) \cdot 2 = \frac{64}{3}$$

7. $I = \iint\limits_A y \, dx \, dy, \quad A:$ trojúhelník určený body $[0,0],[1,1],[2,0]$

Množina A je dána nerovnostmi $\begin{array}{ccccc} 0 & \le & y & \le & 1 \\ y & \le & x & \le & 2-y \end{array}$

$$I = \int_0^1 \left(\int_y^{2-y} y \, dx \right) dy = \int_0^1 [yx]_{x=y}^{x=2-y} \, dy = \int_0^1 (y(2-y)-y^2) \, dy = \int_0^1 (2y-2y^2) \, dy = \left[y^2 - \frac{2y^3}{3} \right]_0^1 = \frac{1}{3}$$

8.

$$\int_0^2 \int_y^{2y} \cos y^2 \, dx \, dy = \int_0^2 \left[x \cos y^2 \right]_{x=y}^{2y} \, dy = \int_0^2 (2y \cos y^2 - y \cos y^2) \, dy = \int_0^2 y \cos y^2 \, dy$$

$$= \left| \begin{array}{cccc} y^2 & = & t & y=0 \Rightarrow t=0 \\ 2y \, dy & = & dt & y=2 \Rightarrow t=4 \end{array} \right| = \int_0^4 \cos t \frac{dt}{2} = \frac{1}{2} [\sin t]_0^4 = \frac{1}{2} \sin 4$$

Integrujte převodem do polárních souřadnic

$$x = u \cos v, y = u \sin v, \quad |J| = u$$

1. $I = \iint\limits_A \sqrt{x^2+y^2} \, dx \, dy, \quad A:$ čtvrtkruh o poloměru r v 1. kvadrantu

Množina $F^{-1}(A)$ je dána nerovnostmi $\begin{array}{ccccc} 0 & \le & u & \le & r \\ 0 & \le & v & \le & \frac{\pi}{2} \end{array}$

$$I = \int_0^{\frac{\pi}{2}} \int_0^r \sqrt{u^2 \cos^2 v + u^2 \sin^2 v} \cdot u \, du \, dv = \int_0^{\frac{\pi}{2}} \int_0^r \sqrt{u^2 (\cos^2 v + \sin^2 v)} \cdot u \, du \, dv = \int_0^{\frac{\pi}{2}} \int_0^r u \cdot u \, du \, dv$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{u^3}{3} \right]_0^r dv = \int_0^{\frac{\pi}{2}} \frac{r^3}{3} \, dv = \frac{r^3}{3} \int_0^{\frac{\pi}{2}} dv = \frac{r^3}{3} \cdot \frac{\pi}{2} = \frac{\pi r^3}{6}$$

2. $I = \iint\limits_A (x^2+y^2) \, dx \, dy, \quad A:$ dána nerovnostmi $1 \le x^2+y^2 \le 4, -x \le y \le x$

Množina $F^{-1}(A)$ je dána nerovnostmi $\begin{array}{ccccc} 1 & \le & u & \le & 2 \\ -\frac{\pi}{4} & \le & v & \le & \frac{\pi}{4} \end{array}$

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\int_1^2 u^2 \cdot u \, du \right) dv = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{u^4}{4} \right]_1^2 dv = \left(\frac{16}{4} - \frac{1}{4} \right) \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} dv = \frac{15}{4} \cdot \frac{\pi}{2}$$

3. $I = \iint\limits_A dx \, dy, \quad A:$ ohraničená křivkami $x^2+y^2 = 2x, x^2+y^2 = 4x, y = \frac{1}{\sqrt{3}}x, y = \sqrt{3}x$

$$\begin{array}{ccc} 2x & \le x^2+y^2 \le & 4x \\ 2u \cos v & \le u^2 \le & 4u \cos v \\ 2 \cos v & \le u \le & 4 \cos v \end{array}$$

$$\operatorname{arctg}\left(\frac{1}{\sqrt{3}}\right) = \operatorname{arctg}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}, \operatorname{arctg} \sqrt{3} = \frac{\pi}{3}$$

Množina $F^{-1}(A)$ je dána nerovnostmi $\begin{array}{ccccc} 2\cos v & \le & u & \le & 4 \cos v \\ \frac{\pi}{6} & \le & v & \le & \frac{\pi}{3} \end{array}$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_{2\cos v}^{4 \cos v} u \, du \, dv = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left[\frac{u^2}{2} \right]_{2 \cos v}^{4 \cos v} dv = \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 12 \cos^2 v \, dv$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 6 \left(\frac{1+\cos 2v}{2} \right) dv = 3 \left[v + \frac{\sin 2v}{2} \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = 3 \left(\frac{\pi}{3} + \frac{\sqrt{3}}{4} - \frac{\pi}{6} - \frac{\sqrt{3}}{4} \right) = \frac{\pi}{2}$$

4. $I = \iint\limits_A dx\,dy, \quad A:$ dána nerovnostmi $x^2+y^2 \le 4, 1 \le y$

$u^2 \le 4, 1 \le u \sin v$,
průnik $x^2+y^2=4, y=1 : x^2=3, x=\pm\sqrt{3}$,
přímka spojující počátek a průsečík $[\pm\sqrt{3},1]$: $y = \pm\frac{1}{\sqrt{3}}$,
$\operatorname{arctg} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$
Množina $F^{-1}(A)$ je dána nerovnostmi $\begin{array}{ccccc} \frac{1}{\sin v} & \le & u & \le & 2 \\ \frac{\pi}{6} & \le & v & \le & \frac{5\pi}{6} \end{array}$

$$I = \int_{\frac{\pi}{6}}^{\frac{5}{6}\pi} \int_{\frac{1}{\sin v}}^{2} u\,du\,dv = \int_{\frac{\pi}{6}}^{\frac{5}{6}\pi} \left[\frac{u^2}{2}\right]_{\frac{1}{\sin v}}^{2} dv = \int_{\frac{\pi}{6}}^{\frac{5}{6}\pi} \left(2 - \frac{1}{2\sin^2 v}\right) dv$$

$$= \frac{1}{2}\left[4v + \operatorname{cotg} v\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} = \frac{4}{3}\pi - \sqrt{3}$$

5. $I = \iint\limits_A dx\,dy, \quad A:$ ohraničená křivkami $x^2+y^2 = x+y, x=0, y=0$

Množina $F^{-1}(A)$ je dána nerovnostmi $\begin{array}{ccccc} 0 & \le & u & \le & \sin v + \cos v \\ 0 & \le & v & \le & \frac{\pi}{2} \end{array}$

$$I = \int_0^{\frac{\pi}{2}} \int_0^{\sin v + \cos v} u\,du\,dv = \frac{1}{2}\int_0^{\frac{\pi}{2}} (\sin v + \cos v)^2\,dv = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 + 2\sin v \cos v)\,dv = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\sin 2v)\,dv$$

$$= \frac{1}{2}\left[v - \frac{\cos 2v}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4} + \frac{1}{4}$$