Nalezněte diferenciál funkce

$$df(x,y) = f_x(x,y)\,dx + f_y(x,y)\,dy, \quad (dx = x - x_0 = h, dy = y - y_0 = k)$$

1. $f(x,y) = x^3y - 2xy^2 + 5x$
   $f_x = 3x^2y - 2y^2 + 5, \quad f_y = x^3 - 4xy$
   $df(x,y) = (3x^2y - 2y^2 + 5)dx + (x^3 - 4xy)dy$

2. $f(x,y) = \mathrm{e}^{x^2-y}, \quad a = (1,0)$
   $f_x = \mathrm{e}^{x^2-y}\,2x, \quad f_y = \mathrm{e}^{x^2-y}(-1)$
   $df(x,y) = (\mathrm{e}^{x^2-y}\,2x)dx - (\mathrm{e}^{x^2-y})dy, \quad df(1,0) = 2\,\mathrm{e}\,dx - \mathrm{e}\,dy$

3. $f(x,y) = \arcsin\frac{x}{\sqrt{x^2+y^2}}, \quad a = (\sqrt{3},1)$
   $f_x = \frac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\frac{\sqrt{x^2+y^2}-x\frac{2x}{2\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{\sqrt{x^2+y^2}}{y}\frac{\frac{y^2}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{y}{x^2+y^2}$
   $f_y = \frac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\left(-\frac{1}{2}\right)\frac{x}{\sqrt{x^2+y^2}^3}2y = -\frac{\sqrt{x^2+y^2}}{y}\frac{xy}{\sqrt{x^2+y^2}^3} = \frac{-x}{x^2+y^2}$
   $df(x,y) = \frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy, \quad df(\sqrt{3},1) = \frac{1}{4}dx - \frac{\sqrt{3}}{4}dy$

Pomocí diferenciálu přibližně vypočtěte

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)dx + f_y(x_0,y_0)dy$$

1. $\sqrt{0,98^2 + 2,05^3}$
   $x_0 = 1, y_0 = 2, dx = -0.02, dy = 0.05$
   $f(x,y) = \sqrt{x^2+y^3}, f(1,2) = 3$
   $f_x = \frac{2x}{2\sqrt{x^2+y^3}} = \frac{x}{\sqrt{x^2+y^3}}, \quad f_y = \frac{3y^2}{2\sqrt{x^2+y^3}}$
   $df(x,y) = \frac{x}{\sqrt{x^2+y^3}}\,dx + \frac{3y^2}{2\sqrt{x^2+y^3}}\,dy, \quad df(1,2) = \frac{1}{3}dx + 2dy$
   $\sqrt{0.98^2 + 2.05^3} \approx 3 + \frac{1}{3}(-0.02) + 2(0.05) = 3 - \frac{2}{300} + \frac{10}{100} = \frac{900-2+30}{300} = \frac{928}{300}$

2. $\mathrm{e}^{0.04^3-0.01}$
   $x_0 = 0, y_0 = 0, dx = 0.04, dy = 0.01$
   $f(x,y) = \mathrm{e}^{x^3-y}, f(0,0) = 1$
   $f_x = \mathrm{e}^{x^3-y}\,3x^2, \quad f_y = \mathrm{e}^{x^3-y}(-1)$
   $df(x,y) = 3x^2\,\mathrm{e}^{x^3-y}\,dx - \mathrm{e}^{x^3-y}\,dy, \quad df(0,0) = 0\,dx - dy$
   $\mathrm{e}^{0.04^3-0.01} \approx 1 - 0.01 = 0.99$

3. $\arcsin\frac{0.47}{1.04}$
   $x_0 = \frac{1}{2}, y_0 = 1, dx = -0.03, dy = 0.04$
   $f(x,y) = \arcsin\frac{x}{y}, \quad f(x_0,y_0) = \arcsin\frac{1}{2} = \frac{\pi}{6}$
   $f_x = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\frac{1}{x} = \frac{y}{x\sqrt{y^2-x^2}}, \quad f_y = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(-\frac{x}{y^2}\right) = -\frac{x}{y\sqrt{y^2-x^2}}$
   $df(x,y) = \frac{y}{x\sqrt{y^2-x^2}}\,dx - \frac{x}{y\sqrt{y^2-x^2}}\,dy, \quad df\left(\frac{1}{2},1\right) = \frac{4}{\sqrt{3}}\,dx - \frac{1}{\sqrt{3}}\,dy$
   $\arcsin\frac{0.47}{1.04} \approx \frac{\pi}{6} + \frac{4}{\sqrt{3}}\cdot(-0.03) - \frac{1}{\sqrt{3}}\cdot 0.04 = \frac{\pi}{6} - \frac{4}{25\sqrt{3}}$

Spočtěte rovnici tečné roviny a normály ke grafu funkce f v bodě (x_0,y_0)

$$\tau: z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-x_0)$$

$$n: \frac{x-x_0}{f_x(x_0,y_0)} = \frac{y-y_0}{f_y(x_0,y_0)} = \frac{z-f(x_0,y_0)}{-1}$$

1. $f(x,y) = x^3 + 3xy^2 - xy + x, \quad (x_0, y_0) = (1,0)$
   $f_x = 3x^2 + 3y^2 - y + 1, \quad f_x(1,0) = 4, \quad f_y = 6xy - x, \quad f_y(1,0) = -1$
   $f(1,0) = 1 + 0 - 0 + 1 = 2$
   $\tau : z = 2 + 4(x-1) - (y-0) \Rightarrow 4x - y - z - 2 = 0$
   parametrické vyjádření normály: $n : [x,y,z] = [1,0,2] + t(4,-1,-1), \quad t \in R$
   obecné vyjádření normály: $\frac{x-1}{4} = \frac{y}{-1} = \frac{z-2}{-1}$

2. $f(x,y) = \ln(x+2y), \quad (x_0,y_0) = (2,1)$
   $f_x = \frac{1}{x+2y}, \quad f_x(2,1) = \frac{1}{4}, \quad f_y = \frac{2}{x+2y}, \quad f_y(2,1) = \frac{1}{2}$
   $f(2,1) = \ln 4$
   $\tau : z = \ln 4 + \frac{1}{4}(x-2) + \frac{1}{2}(y-1) \Rightarrow x + 2y - 4z + 4(\ln 4 - 1) = 0$
   $n : \frac{x-2}{\frac{1}{4}} = \frac{y-1}{\frac{1}{2}} = \frac{z - \ln 4}{-1}$
   $n : 4(x-2) = 2(y-1) = \ln 4 - z$
   $n : [x,y,z] = [2,1,\ln 4] + t(1,2,-4), \quad t \in R$

Pomocí gradientu vypočtěte derivaci funkce f v bodě a ve směru $\vec{u} = (u_1, u_2)$

$$\nabla(x,y) = (f_x(x,y), f_y(x,y)), \quad f_u = \nabla(x,y) \cdot \vec{u} = f_x(x,y)u_1 + f_y(x,y)u_2$$

1. $f(x,y) = \ln(x^2+y^2), \quad a = (5,-2), \quad \vec{u} = (-5,2)$
   $f_x = \frac{2x}{x^2+y^2}, f_y = \frac{2y}{x^2+y^2}, \quad \nabla(x,y) = \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}\right), \nabla(5,-2) = \left(\frac{10}{29}, \frac{-4}{29}\right)$
   $f_u(5,-2) = \nabla(-5,2) \cdot \vec{u} = \left(\frac{10}{29}, \frac{-4}{29}\right) \cdot (-5,2) = -\frac{50}{29} - \frac{8}{29} = -\frac{58}{29} = -2$

2. $f(x,y) = \operatorname{arctg}(xy), \quad a = (1,1), \quad \vec{u} = (\sqrt{2}, \sqrt{2})$
   $f_x = \frac{y}{1+x^2y^2}, f_y = \frac{x}{1+x^2y^2}, \quad \nabla(x,y) = \left(\frac{y}{1+x^2y^2}, \frac{x}{1+x^2y^2}\right), \nabla(1,1) = \left(\frac{1}{2}, \frac{1}{2}\right)$
   $f_u(a) = \nabla(1,1) \cdot \vec{u} = \left(\frac{1}{2}, \frac{1}{2}\right) \cdot (\sqrt{2}, \sqrt{2}) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

3. $f(x,y,z) = xy^2 - z^3 + xyz, \quad a = (-1,1,2), \quad \vec{u} = (1,0,-1)$
   $f_x = y^2 + yz, f_y = 2xy + xz, f_z = -3z^2 + xy$
   $\nabla(x,y,z) = (y^2 + yz, 2xy + xz, -3z^2 + xy), \nabla(-1,1,2) = (3,-4,-13)$
   $f_u(a) = (3,-4,-13) \cdot (1,0,-1) = 3 + 13 = 16$

Nalezněte diferenciál druhého řádu funkce f

$$d^2 f(x,y) = f_{xx} dx^2 + 2 f_{xy} dx dy + f_{yy} dy^2$$

1. $f(x,y) = x^3 y - xy^3$
   $f_x = 3x^2 y - y^3, \quad f_y = x^3 - 3xy^2$
   $f_{xx} = 6xy, \quad f_{yy} = -6xy, \quad f_{xy} = 3x^2 - 3y^2$
   $d^2 f(x,y) = 6xy\, dx^2 + (6x^2 - 6y^2)\, dx\, dy - 6xy\, dy^2$

2. $f(x,y) = \mathrm{e}^y \cos x, (x_0,y_0) = (0,0)$
   $f_x = \mathrm{e}^y(-\sin x), \quad f_y = \mathrm{e}^y \cos x$
   $f_{xx} = -\mathrm{e}^y \cos x, \quad f_{yy} = \mathrm{e}^y \cos x, \quad f_{xy} = -\mathrm{e}^y \sin x$
   $d^2 f(x,y) = -\mathrm{e}^y \cos x\, dx^2 - 2\,\mathrm{e}^y \sin x\, dx\, dy + \mathrm{e}^y \cos x\, dy^2$
   $d^2 f(x_0,y_0) = -1\, dx^2 - 0\, dx\, dy + 1\, dy^2$

3. $f(x,y,z) = xyz$
   $f_x = yz, \quad f_y = xz, \quad f_z = xy$
   $f_{xx} = 0, \quad f_{xy} = z, \quad f_{yy} = 0, \quad f_{yz} = x, \quad f_{zz} = 0, \quad f_{xz} = y$
   $d^2 f(x,y,z) = 0\, dx^2 + 0\, dy^2 + 0\, dz^2 + 2(z\, dx\, dy + x\, dy\, dz + y\, dx\, dz)$

Nalezněte Taylorův polynom řádu k funkce f v bodě (x_0, y_0)

$$T_k(x,y) = f(x_0,y_0) + \frac{1}{1!}df(x_0,y_0) + \frac{1}{2!}d^2f(x_0,y_0) + ... + \frac{1}{k!}d^kf(x_0,y_0)$$

$$d^kf(x,y) = \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} \frac{\partial^k f}{\partial x^{k-j}\partial y^j}(x,y)dx^{k-j}dy^j, \quad dx = x - x_0, dy = y - y_0$$

1. $f(x,y) = \sqrt{x^2+y^2}, \quad (x_0,y_0) = (4,3), \quad k = 2$
   $f(4,3) = \sqrt{16+9} = 5$
   $f_x = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}, \quad f_x(4,3) = \frac{4}{5}$
   $f_y = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}, \quad f_y(4,3) = \frac{3}{5}$
   $f_{xx} = \frac{\sqrt{x^2+y^2} - x\frac{2x}{2\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{y^2}{\sqrt{x^2+y^2}^3}, \quad f_{xx}(4,3) = \frac{9}{125}$
   $f_{yy} = \frac{\sqrt{x^2+y^2} - y\frac{2y}{2\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{x^2}{\sqrt{x^2+y^2}^3}, f_{xx}(4,3) = \frac{16}{125}$
   $f_{xy} = -\frac{1}{2}\frac{x}{\sqrt{x^2+y^2}^3}2y = \frac{-xy}{\sqrt{x^2+y^2}^3}, \quad f_{x,y} = -\frac{12}{125}$
   $$T_2(x,y) = 5 + \frac{4}{5}(x-4) + \frac{3}{5}(y-3) + \frac{1}{2}\left(\frac{9}{125}(x-4)^2 - 2\frac{12}{125}(x-4)(y-3) + \frac{16}{125}(y-3)^2\right)$$
   $$= 5 + \frac{4}{5}(x-4) + \frac{3}{5}(y-3) + \frac{9}{250}(x-4)^2 - \frac{12}{125}(x-4)(y-3) + \frac{8}{125}(y-3)^2$$

2. $f(x,y) = \sin(x^2+y), \quad (x_0,y_0) = (0, \frac{\pi}{4}), \quad k = 2$

   $f(0, \frac{\pi}{4}) = \frac{\sqrt{2}}{2}$
   $f_x = \cos(x^2+y)2x, f_x(0, \frac{\pi}{4}) = 0$
   $f_y = \cos(x^2+y), f_y(0, \frac{\pi}{4}) = \frac{\sqrt{2}}{2}$
   $f_{yy} = -\sin(x^2+y), f_{yy}(0, \frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$
   $f_{xy} = -\sin(x^2+y)2x, f_{xy}(0, \frac{\pi}{4}) = 0$
   $f_{xx} = -\sin(x^2+y)2x + \cos(x^2+y)2,$
   $f_{xx}(0, \frac{\pi}{4}) = \sqrt{2}$

   $$T_2(x,y) = \frac{\sqrt{2}}{2} + 0x + \frac{\sqrt{2}}{2}\left(y - \frac{\pi}{4}\right) + \frac{1}{2}\left(\sqrt{2}x^2 + 2\cdot 0x\left(y-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{2}\left(y-\frac{\pi}{4}\right)^2\right)$$
   $$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(y - \frac{\pi}{4}\right) + \frac{1}{2}\left(\sqrt{2}x^2 - \frac{\sqrt{2}}{2}\left(y-\frac{\pi}{4}\right)^2\right)$$

3. $f(x,y) = \ln(1+x)\ln(1+y), \quad (x_0,y_0) = (0,0), \quad k = 3$

   $f(0,0) = 0$
   $f_x = \frac{1}{1+x}\ln(1+y), f_x(0,0) = 0$
   $f_y = \ln(1+x)\frac{1}{1+y}, f_y(0,0) = 0$
   $f_{xx} = -\frac{1}{(1+x)^2}\ln(1+y), f_{xx}(0,0) = 0$
   $f_{xy} = \frac{1}{1+x}\frac{1}{1+y}, f_{xy}(0,0) = 1$
   $f_{yy} = -\ln(1+x)\frac{1}{(1+y)^2}$
   $f_{xxx} = 2\frac{1}{(1+x)^3}\ln(1+y), f_{xxx}(0,0) = 0$
   $f_{xxy} = -\frac{1}{(1+x)^2}\frac{1}{1+y}, f_{xxy}(0,0) = -1$
   $f_{yyy} = 2\ln(1+x)\frac{1}{(1+y)^3}, f_{yyy}(0,0) = 0$
   $f_{yyx} = -\frac{1}{1+x}\frac{1}{(1+y)^2}, f_{yyx}(0,0) = -1$

   $$T_3(x,y) = 0 + 0\,x + 0\,y + \frac{1}{2}(0\,x^2 + 2\cdot 1\,x\,y + 0\,y^2) + \frac{1}{6}(0\,x^3 + 3(-1)\,x^2\,y + 3(-1)\,x\,y^2 + 0\,y^3)$$
   $$= xy - \frac{1}{2}x^2y - \frac{1}{2}xy^2$$