Vzorce na integrování

1. $\int x^s\,dx = \frac{x^{s+1}}{s+1} + C,\ s \neq -1$
2. $\int \frac{1}{x}\,dx = \ln|x| + C$
3. $\int e^x\,dx = e^x + C$
4. $\int a^x\,dx = \frac{a^x}{\ln a} + C$
5. $\int \sin x\,dx = -\cos x + C$
6. $\int \cos x\,dx = \sin x + C$
7. $\int \frac{1}{\cos^2 x}\,dx = \operatorname{tg} x + C$
8. $\int \frac{1}{\sin^2 x}\,dx = -\operatorname{cotg} x + C$
9. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C = -\arccos x + K$
10. $\int \frac{1}{1+x^2}\,dx = \operatorname{arctg} x + C = -\operatorname{arccotg} x + K$
11. $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C$
12. $\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln(x + \sqrt{1+x^2}) + C$
13. $\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln|x + \sqrt{x^2-1}| + C$
14. $\int \sinh x\,dx = \cosh x + C$
15. $\int \cosh x\,dx = \sinh x + C$

$$\int [af(x) + bg(x)]\,dx = a\int f(x)\,dx + b\int g(x)\,dx$$

Pozor !!!
Neplatí žádné univerzální pravidlo pro integrál součinu a podílu dvou funkcí!!!

Integrujte

1. $\int(3x^3-x^2-3x+5)\,dx = 3\int x^3\,dx - \int x^2\,dx - 3\int x\,dx + 5\int dx = 3\cdot\frac{x^4}{4} - \frac{x^3}{3} - 3\cdot\frac{x^2}{2} + 5\cdot x + C$

2. $\int(x+1)(x-1)(x^2-1)\,dx = \int(x^2-1)(x^2-1)\,dx = \int(x^4-2x^2+1)\,dx = \frac{x^5}{5} - 2\frac{x^3}{3} + x + C$

3. $\int \frac{x^5+4x^4-x^2+2}{x^2}\,dx = \int(x^3+4x^2-1+\frac{2}{x^2})\,dx = \frac{x^4}{4} + 4\frac{x^3}{3} - x - 2\frac{1}{x} + C$

4. $\int \frac{(x-1)^2}{x^3}\,dx = \int \frac{x^2-2x+1}{x^3}\,dx = \int \left(\frac{1}{x} - \frac{2}{x^2} + \frac{1}{x^3}\right)\,dx = \ln|x| + \frac{2}{x} - \frac{1}{2x^2} + C$

5. $\int \frac{\sqrt[3]{x^2}-\sqrt[4]{x}}{\sqrt{x}}\,dx = \int \left(\frac{x^{\frac{2}{3}}}{x^{\frac{1}{2}}} - \frac{x^{\frac{1}{4}}}{x^{\frac{1}{2}}}\right)\,dx = \int(x^{\frac{1}{6}} - x^{-\frac{1}{4}})\,dx = \frac{x^{\frac{7}{6}}}{\frac{7}{6}} - \frac{x^{\frac{3}{4}}}{\frac{3}{4}} + C = \frac{6\sqrt[6]{x^7}}{7} - \frac{4\sqrt[4]{x^3}}{3} + C =$ $\frac{6x\sqrt[6]{x}}{7} - \frac{4\sqrt[4]{x^3}}{3} + C$

6. $\int \frac{1}{\sqrt{3-3x^2}}\,dx = \int \frac{1}{\sqrt{3(1-x^2)}}\,dx = \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{1-x^2}}\,dx = \frac{1}{\sqrt{3}}\arcsin x + C = \frac{\sqrt{3}}{3}\arcsin x + C$

7. $\int \frac{3\cdot 2^x - 2\cdot 3^x}{2^x}\,dx = \int(3-2\left(\frac{3}{2}\right)^x)\,dx = 3x - 2\cdot\left(\frac{3}{2}\right)^x \frac{1}{\ln\frac{3}{2}} + C$

8. $\int \frac{1+\cos^2 x}{1+\cos 2x}\,dx = \int \frac{1+\cos^2 x}{1+\cos^2 x-\sin^2 x}\,dx = \int \frac{1+\cos^2 x}{2\cos^2 x}\,dx = \int \frac{1}{2\cos^2 x}\,dx + \int \frac{\cos^2 x}{2\cos^2 x}\,dx = \frac{1}{2}\operatorname{tg} x + \frac{1}{2}x + C$

9. $\int 2\sin^2\frac{x}{2}\,dx = \int(\sin^2\frac{x}{2}+\sin^2\frac{x}{2})\,dx = \int(\sin^2\frac{x}{2}+1-\cos^2\frac{x}{2})\,dx = \int(1-(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}))\,dx = \int(1-\cos x)\,dx = x - \sin x + C$

Integrace metodou per partes

1.
$$\int x\sin x\,dx = \left|\begin{array}{ll} u=x & u'=1 \\ v'=\sin x & v=-\cos x\end{array}\right| = x\cdot(-\cos x) - \int 1\cdot(-\cos x)\,dx = -x\cos x + \sin x + C$$

2.
$$\int x^2e^x\,dx = \left|\begin{array}{ll} u=x^2 & u'=2x \\ v'=e^x & v=e^x\end{array}\right| = x^2e^x - \int 2xe^x\,dx = x^2e^x - 2\int xe^x\,dx = \left|\begin{array}{ll} u=x & u'=1 \\ v'=e^x & v=e^x\end{array}\right|$$
$$= x^2e^x - 2[xe^x - \int 1e^x\,dx] = x^2e^x - 2xe^x + 2e^x + C = e^x(x^2-2x+2) + C$$

3.
$$\int \ln x\,dx = \int 1\cdot\ln x\,dx = \left|\begin{array}{ll} u=\ln x & u'=\frac{1}{x} \\ v'=1 & v=x\end{array}\right| = x\ln x - \int \frac{1}{x}\cdot x\,dx = x\ln x - \int dx = x\ln x - x + C$$

4.
$$\int x\operatorname{arctg} x\,dx = \left|\begin{array}{ll} u=\operatorname{arctg} x & u'=\frac{1}{1+x^2} \\ v'=x & v=\frac{x^2}{2}\end{array}\right| = \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx = \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}\,dx$$
$$= \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)\,dx = \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}x + \frac{1}{2}\operatorname{arctg} x + C$$

5.
$$\int e^x\cos x\,dx = \left|\begin{array}{ll} u=e^x & u'=e^x \\ v'=\cos x & v=\sin x\end{array}\right| = e^x\sin x - \int e^x\sin x\,dx = \left|\begin{array}{ll} u=e^x & u'=e^x \\ v'=\sin x & v=-\cos x\end{array}\right|$$
$$= e^x\sin x - [e^x(-\cos x) - \int e^x(-\cos x)\,dx] = e^x\sin x + e^x\cos x - \int e^x\cos x\,dx$$
$$2\int e^x\cos x\,dx = e^x\sin x + e^x\cos x$$
$$\int e^x\cos x\,dx = \frac{1}{2}(e^x\sin x + e^x\cos x) + C$$

6.
$$\int \sin^2 x\,dx = \int \sin x \cdot \sin x\,dx = \left| \begin{array}{ll} u = \sin x & u' = \cos x \\ v' = \sin x & v = -\cos x \end{array} \right|$$

$$= -\sin x \cos x + \int \cos^2 x\,dx = -\sin x \cos x + \int (1 - \sin^2 x)\,dx = -\sin x \cos x + \int dx - \int \sin^2 x\,dx$$

Odtud

$$2\int \sin^2 x\,dx = -\sin x \cos x + x$$

$$\int \sin^2 x\,dx = \frac{1}{2}(x - \sin x \cos x) + C$$

Substituční metoda

1.
$$\int \frac{1}{1-x}\,dx = \left| \begin{array}{lcl} 1-x & = & t \\ -dx & = & dt \end{array} \right| = \int \frac{1}{t}\,\frac{dt}{-1} = -\int \frac{1}{t}\,dt = -\ln|t| + C = -\ln|1-x| + C$$

2.
$$\int \frac{e^x}{2+e^x}\,dx = \left| \begin{array}{lcl} 2+e^x & = & t \\ e^x dx & = & dt \end{array} \right| = \int \frac{e^x}{t}\,\frac{dt}{e^x} = \int \frac{1}{t}\,dt = \ln|t| + C = \ln|2+e^x| + C = \ln(2+e^x) + C$$

3.
$$\int \operatorname{cotg} x\,dx = \int \frac{\cos x}{\sin x}\,dx = \left| \begin{array}{lcl} \sin x & = & t \\ \cos x dx & = & dt \end{array} \right| = \int \frac{1}{t}\,dt = \ln|t| + C = \ln|\sin x| + C$$

4.
$$\int \frac{2x}{\sqrt{x^2+1}}\,dx = \left| \begin{array}{lcl} x^2+1 & = & t \\ 2x dx & = & dt \end{array} \right| = \int \frac{1}{\sqrt{t}}\,dt = \int t^{-\frac{1}{2}}\,dt = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{t} + C = 2\sqrt{x^2+1} + C$$

5.
$$\int \sqrt{-2x+8}\,dx = \left| \begin{array}{lcl} -2x+8 & = & t \\ -2dx & = & dt \end{array} \right| = \int \sqrt{t}\,\frac{dt}{-2} = -\frac{1}{2}\int t^{\frac{1}{2}}\,dt = -\frac{1}{2}\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= -3\sqrt{t^3} + C = -3\sqrt{(-2x+8)^3} + C$$

6.
$$\int \frac{\sqrt{\ln x}}{x}\,dx = \left| \begin{array}{lcl} \ln x & = & t \\ \frac{1}{x} dx & = & dt \end{array} \right| = \int \sqrt{t}\,dt = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}\sqrt{\ln^3 x} + C$$

7.
$$\int \frac{\operatorname{arctg}^2 x}{1+x^2}\,dx = \left| \begin{array}{lcl} \operatorname{arctg} x & = & t \\ \frac{1}{1+x^2} dx & = & dt \end{array} \right| = \int t^2\,dt = \frac{t^3}{3} + C = \frac{\operatorname{arctg}^3 x}{3} + C$$

8.
$$\int \frac{1}{2x-1}\,dx = \left| \begin{array}{lcl} 2x-1 & = & t \\ 2dx & = & dt \end{array} \right| = \int \frac{1}{t}\,\frac{dt}{2} = \frac{1}{2}\int \frac{1}{t}\,dt = \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln|2x-1| + C$$

9.
$$\int e^{-x^3} x^2\,dx = \left| \begin{array}{lcl} -x^3 & = & t \\ -3x^2 dx & = & dt \end{array} \right| = \int e^t x^2 \frac{dt}{-3x^2} = -\frac{1}{3}\int e^t\,dt = -\frac{1}{3}e^t + C = -\frac{1}{3}e^{-x^3} + C$$

10.
$$\int \cos^3 x \sin x\,dx = \left| \begin{array}{lcl} \cos x & = & t \\ -\sin x dx & = & dt \end{array} \right| = -\int t^3\,dt = -\frac{t^4}{4} + C = -\frac{\cos^4 x}{4} + C$$

11.

$$\int \frac{1}{1+9x^2}\,dx = \int \frac{1}{1+(3x)^2}\,dx = \left| \begin{array}{ccc} 3x & = & t \\ 3dx & = & dt \end{array} \right| = \int \frac{1}{1+t^2}\frac{dt}{3} = \frac{1}{3}\operatorname{arctg} t + C = \frac{1}{3}\operatorname{arctg} 3x + C$$

12.

$$\int \frac{1}{\sqrt{4-9x^2}}\,dx = \int \frac{1}{\sqrt{4(1-\frac{9x^2}{4})}}\,dx == \int \frac{1}{2\sqrt{1-\left(\frac{3x}{2}\right)^2}}\,dx = \left| \begin{array}{ccc} \frac{3x}{2} & = & t \\ \frac{3}{2}dx & = & dt \end{array} \right| = \frac{1}{2}\int \frac{1}{\sqrt{1-t^2}}\frac{2dt}{3}$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{1-t^2}}\,dt = \frac{1}{3}\arcsin t + C = \frac{1}{3}\arcsin\frac{3x}{2} + C$$

13.

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a^2}\int \frac{1}{1+\frac{x^2}{a^2}}\,dx = \left| \begin{array}{ccc} \frac{x}{a} & = & t \\ \frac{1}{a}dx & = & dt \end{array} \right| = \frac{1}{a^2}\int \frac{1}{1+t^2}\cdot a\,dt = \frac{1}{a}\operatorname{arctg} t + C = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C$$

Převeďte na součet polynomu a ryze racionální funkce a integrujte

1.

$$\int \frac{x}{2x+1}\,dx = |\text{dělením}| = \int \left(\frac{1}{2} - \frac{1}{2}\frac{1}{2x+1}\right)dx = \frac{1}{2}x - \frac{1}{2}\int \frac{1}{2x+1}\,dx = \left| \begin{array}{ccc} 2x+1 & = & t \\ 2dx & = & dt \end{array} \right|$$

$$= \frac{1}{2}x - \frac{1}{2}\int \frac{1}{t}\frac{dt}{2} = \frac{1}{2}x - \frac{1}{4}\int \frac{1}{t}\,dt = \frac{1}{2}x - \frac{1}{4}\ln|t| + C = \frac{1}{2}x - \frac{1}{4}\ln|2x+1| + C$$

2.

$$\int \frac{x^2-1}{x^2+1}\,dx = \int \left(1 - \frac{2}{x^2+1}\right)dx = x - 2\operatorname{arctg} x + C$$

3.

$$\int \frac{x^4}{1-x}\,dx = \int \left(-x^3 - x^2 - x - 1 + \frac{1}{1-x}\right) = \left| \begin{array}{ccc} 1-x & = & t \\ -dx & = & dt \end{array} \right| = -\frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2} - x - \int \frac{1}{t}\,dt$$

$$= -\frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2} - x - \ln|1-x| + C$$

Rozložte ryze lomenou funkci na součet parciálních zlomků

1. $R(x) = \frac{2x+1}{x^3-x}$
   Rozložíme jmenovatel na součin kořenových faktorů (v reálném oboru).
   $x^3 - x = x(x^2-1) = x(x-1)(x+1)$
   Protože máme tři různé jednonásobné reálné kořeny, rozklad dopadne takto:

$$\frac{2x+1}{x^3-x} = \frac{2x+1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

   Vynásobíme rovnost společným jmenovatelem z levé strany.

$$2x+1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

   Máme dvě možnosti
   a) do rovnice postupně dosadíme reálné kořeny původního jmenovatele:

$$\begin{array}{lccc} x=0: & 0+1 = A(-1)(1) + B\cdot 0(1) + C\cdot 0(-1) & \Rightarrow & A=-1 \\ x=1: & 2+1 = B\cdot 1(1+1) & \Rightarrow & B=\frac{3}{2} \\ x=-1: & -2+1 = C(-1)(-2) & \Rightarrow & C=-\frac{1}{2} \end{array}$$

b) rovnici ještě upravíme a porovnáme koeficienty u jednotlivých mocnin x na pravé a na levé straně:

$$2x + 1 = Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx$$
$$2x + 1 = x^2(A + B + C) + x(B - C) - A$$
$$\begin{array}{lrl} x^2: & 0 = & A + B + C \\ x^1: & 2 = & B - C \\ x^0: & 1 = & -A \end{array}$$

Řešením soustavy tří rovnic o třech neznámých dostaneme $A = -1$, $B = \frac{3}{2}$, $C = -\frac{1}{2}$.
Celkem máme:

$$R(x) = \frac{-1}{x} + \frac{\frac{3}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}.$$

2. $R(x) = \frac{x^2-x-1}{x^3-2x^2+x}$

$$x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x-1)^2$$

Máme jeden jednonásobný kořen $x = 0$ a dvojnásobný kořen $x = 1$. Rozklad bude vypadat takto:

$$\frac{x^2 - x - 1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$
$$x^2 - x - 1 = A(x-1)^2 + Bx(x-1) + Cx$$
$$x^2 - x - 1 = Ax^2 - 2Ax + A + Bx^2 - Bx + Cx$$

Koeficienty najdeme kombinací obou metod. Nejprve dosadíme dva reálné kořeny a potom poslední koeficient získáme medotou porovnávání koeficientů.

$$x = 0: \quad -1 = A \Rightarrow A = -1$$
$$x = 1: \quad -1 = C \Rightarrow C = -1$$

Porovnáme koeficient u x^1:

$$-1 = -2A - B + C \Rightarrow B = 2$$

A tedy

$$R(x) = -\frac{1}{x} + \frac{2}{x-1} - \frac{1}{(x-1)^2}.$$

3. $R(x) = \frac{2x^3+x+2}{x^4+x^3+x^2}$

$$x^4 + x^3 + x^2 = x^2(x^2 + x + 1)$$

Máme dvojnásobný reálný kořen $x = 0$ a kvadratický dvojčlen se záporným diskriminantem.

$$\frac{2x^3 + x + 2}{x^2(x^2 + x + 1)} = \frac{A}{x^2} + \frac{B}{x} + \frac{Cx + D}{x^2 + x + 1}$$
$$2x^3 + x + 2 = A(x^2 + x + 1) + Bx(x^2 + x + 1) + (Cx + D)x^2$$
$$2x^3 + x + 2 = Ax^2 + Ax + A + Bx^3 + Bx^2 + Bx + Cx^3 + Dx^2$$
$$2x^3 + x + 2 = x^3(B + C) + x^2(A + B + D) + x(A + B) + A$$

Porovnáním koeficientů:

$$\begin{array}{ccccccccc} & & B & + & C & & & = & 2, \\ A & + & B & & & + & D & = & 0, \\ A & + & B & & & & & = & 1, \\ A & & & & & & & = & 2. \end{array}$$

Odtud $A = 2, B = -1, C = 3, D = -1$, tj.

$$R(x) = \frac{2}{x^2} - \frac{1}{x} + \frac{3x - 1}{x^2 + x + 1}.$$

Integrování parciálních zlomků

I. typ: $\int \frac{A}{x-\alpha}\,dx$

- $$\int \frac{3}{x-5}\,dx = \left| \begin{array}{rcl} x-5 & = & t \\ dx & = & dt \end{array} \right| = 3\int \frac{1}{t}\,dt = 3\ln|x-5| + C$$

II. typ:$\int \frac{A}{(x-\alpha)^k}\,dx, \quad k \neq -1$

- $$\int \frac{\sqrt{2}}{(x+2)^5}\,dx = \left| \begin{array}{rcl} x+2 & = & t \\ dx & = & dt \end{array} \right| = \sqrt{2}\int \frac{1}{t^5}\,dt = \sqrt{2}\int t^{-5}\,dt = \frac{\sqrt{2}}{-4}\frac{1}{t^4} + C = -\frac{\sqrt{2}}{4}\frac{1}{(x+2)^4} + C$$

III. typ: $\int \frac{Bx+C}{x^2+px+q}\,dx$, $p^2-4q<0$

- $$\int \frac{2x-3}{x^2-3x+5}\,dx = \left| \begin{array}{rcl} x^2-3x+5 & = & t \\ (2x-3)dx & = & dt \end{array} \right| = \int \frac{1}{t}\,dt = \ln|t| + C = \ln|x^2-3x+5| + C$$

- $$\int \frac{1}{x^2-2x+3}\,dx = \int \frac{1}{x^2-2x+1-1+3}\,dx = \int \frac{1}{(x-1)^2+2}\,dx = \int \frac{1}{2[\frac{(x-1)^2}{2}+1]}\,dx$$
$$= \frac{1}{2}\int \frac{1}{\left(\frac{x-1}{\sqrt{2}}\right)^2+1}\,dx = \left| \begin{array}{rcl} \frac{x-1}{\sqrt{2}} & = & t \\ \frac{1}{\sqrt{2}}dx & = & dt \end{array} \right| = \frac{1}{2}\int \frac{1}{t^2+1}\sqrt{2}dt = \frac{\sqrt{2}}{2}\int \frac{1}{t^2+1}\,dt$$
$$= \frac{\sqrt{2}}{2}\operatorname{arctg} t + C = \frac{\sqrt{2}}{2}\operatorname{arctg}\left(\frac{x-1}{\sqrt{2}}\right) + C$$

- $$\int \frac{3x-1}{x^2+x+1}\,dx = 3\int \frac{x-\frac{1}{3}}{x^2+x+1}\,dx = \frac{3}{2}\int \frac{2x-\frac{2}{3}}{x^2+x+1}\,dx = \frac{3}{2}\int \frac{2x+1-1-\frac{2}{3}}{x^2+x+1}\,dx$$
$$= \frac{3}{2}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{3}{2}\int \frac{\frac{5}{3}}{x^2+x+1}\,dx = \frac{3}{2}\ln|x^2+x+1| - \frac{5}{2}\int \frac{1}{x^2+x+1}\,dx$$
$$\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{x^2+x+\frac{1}{4}-\frac{1}{4}+1}\,dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

Máme dvě možnosti

$$a) = \frac{4}{3}\int \frac{1}{\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2+1}\,dx = \frac{4}{3}\int \frac{1}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}\,dx = \left| \begin{array}{rcl} \frac{2x+1}{\sqrt{3}} & = & t \\ \frac{2}{\sqrt{3}}dx & = & dt \end{array} \right| = \frac{4}{3}\int \frac{1}{t^2+1}\frac{\sqrt{3}}{2}\,dt$$
$$= \frac{4}{3}\frac{\sqrt{3}}{2}\operatorname{arctg} t + C = \frac{2\sqrt{3}}{3}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$$b) = \left| \begin{array}{rcl} x+\frac{1}{2} & = & t \\ dx & = & dt \end{array} \right| = \frac{3}{4}\int \frac{1}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dt = \left|\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C\right| = \frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

Celkem máme

$$\int \frac{3x-1}{x^2+x+1}\,dx = \frac{3}{2}\ln|x^2+x+1| - \frac{5}{2}\frac{2\sqrt{3}}{3}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$
$$= \frac{3}{2}\ln|x^2+x+1| - \frac{5\sqrt{3}}{3}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

Ryze lomenou funkci převeďte na součet parciálních zlomků a integrujte

1.

$$\int \frac{2x+1}{x^3-x}\,dx = \int \left(\frac{-1}{x} + \frac{\frac{3}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1} \right) dx = -\ln|x| + \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

2.

$$\int \frac{x^2-x-1}{x^3-2x^2+x}\,dx = \int \left(-\frac{1}{x} + \frac{2}{x-1} - \frac{1}{(x-1)^2} \right) dx = -\ln|x| + 2\ln|x-1| + \frac{1}{x-1} + C$$

3.

$$\int \frac{2x^3+x+2}{x^4+x^3+x^2}\,dx = \int \left(\frac{2}{x^2} - \frac{1}{x} + \frac{3x-1}{x^2+x+1} \right) dx$$

$$= -2\frac{1}{x} - \ln|x| + \frac{3}{2}\ln|x^2+x+1| - \frac{5\sqrt{3}}{3}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

(viz dřív)

4.

$$\int \frac{x^2+2x}{x^4+2x^2-3}\,dx$$

Rozložíme na parciální zlomky:

$$\frac{x^2+2x}{x^4+2x^2-3} = |x^2=y| = \frac{x^2+2x}{y^2+2y-3} = \frac{x^2+2x}{(y-1)(y+3)} = \frac{x^2+2x}{(x^2-1)(x^2+3)} = \frac{x^2+2x}{(x-1)(x+1)(x^2+3)}$$

$$\frac{x^2+2x}{(x-1)(x+1)(x^2+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+3}$$

$$x^2+2x = A(x+1)(x^2+3) + B(x-1)(x^2+3) + (Cx+D)(x-1)(x+1)$$

$$\begin{array}{llll} x=1: & 1+2=8A & \Rightarrow & A=\frac{3}{8} \\ x=-1: & 1-2=-8B & \Rightarrow & B=\frac{1}{8} \\ x^0: & 0=3A-3B-D & \Rightarrow & D=\frac{3}{4} \\ x^3: & 0=A+B+C & \Rightarrow & C=-\frac{1}{2} \end{array}$$

$$\frac{x^2+2x}{(x-1)(x+1)(x^2+3)} = \frac{\frac{3}{8}}{x-1} + \frac{\frac{1}{8}}{x+1} + \frac{-\frac{1}{2}x+\frac{3}{4}}{x^2+3}$$

$$\int \frac{x^2+2x}{x^4+2x^2-3}\,dx = \int \left(\frac{3}{8}\frac{1}{x-1} + \frac{1}{8}\frac{1}{x+1} - \frac{1}{4}\frac{2x-3}{x^2+3} \right) dx = \frac{3}{8}\ln|x-1| + \frac{1}{8}\ln|x+1| - \frac{1}{4}\int \frac{2x-3}{x^2+3}\,dx$$

$$= \frac{3}{8}\ln|x-1| + \frac{1}{8}\ln|x+1| - \frac{1}{4}\int \left(\frac{2x}{x^2+3} - \frac{3}{x^2+3} \right) dx$$

$$= \frac{3}{8}\ln|x-1| + \frac{1}{8}\ln|x+1| - \frac{1}{4}\ln(x^2+3) + \frac{3}{4}\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x}{\sqrt{3}} + C$$

5.

$$\int \frac{x^4}{x^3-1}\,dx = \left| x^4 : (x^3-1) = x + \frac{x}{x^3-1} \right| = \int \left(x + \frac{x}{x^3-1} \right) dx$$

Ryze lomenou funkci $\frac{x}{x^3-1}$ rozložíme na součet parciálních zlomků

$$\frac{x}{x^3-1} = \frac{x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$x = A(x^2+x+1) + (Bx+C)(x-1)$$

$$\begin{array}{llll} x^2: & 0=A+B & \Rightarrow & A=-B \\ x^0: & 0=A-C & \Rightarrow & A=C \\ x^1: & 1=A-B+C & \Rightarrow & 1=A+A+A \end{array}$$

Řešením soustavy jsou koeficienty $A = \frac{1}{3}, B = -\frac{1}{3}, C = \frac{1}{3}$. Pokračujeme v integrování

$$\int \frac{x^4}{x^3-1}\,dx = \int \left(x + \frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+x+1} \right) dx$$

integrál $\int \frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+x+1}\,dx$ spočítáme zvlášť:

$$\int \frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+x+1}\,dx = -\frac{1}{3}\int \frac{x-1}{x^2+x+1}\,dx = -\frac{1}{6}\int \frac{2x-2}{x^2+x+1}\,dx = -\frac{1}{6}\int \frac{2x+1-1-2}{x^2+x+1}\,dx$$

$$= -\frac{1}{6}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{1}{6}\int \frac{-3}{x^2+x+1}\,dx = -\frac{1}{6}\ln|x^2+x+1| + \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx$$

$$= -\frac{1}{6}\ln|x^2+x+1| + \frac{1}{2}\int \frac{1}{(x^2+x+\frac{1}{4})+\frac{3}{4}}\,dx = -\frac{1}{6}\ln|x^2+x+1| + \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$= \begin{vmatrix} x+\frac{1}{2} & = & t \\ dx & = & dt \end{vmatrix} = -\frac{1}{6}\ln|x^2+x+1| + \frac{1}{2}\int \frac{1}{t^2+\frac{3}{4}}\,dt = -\frac{1}{6}\ln|x^2+x+1| + \frac{1}{2}\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2t}{\sqrt{3}} + C$$

$$= -\frac{1}{6}\ln|x^2+x+1| + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + C$$

Celkem

$$\int \frac{x^4}{x^3-1}\,dx = \int \left(x + \frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+x+1} \right) dx$$

$$= \frac{x^2}{2} + \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + C$$

Některé integrály typu $\int \sin^n x \cos^m x\,dx$

1.

$$\int \sin^3 x \cos^4 x\,dx = \int \sin^2 x \cos^4 x \sin x\,\mathrm{dx} = \int (1-\cos^2 x)\cos^4 x \sin x\,dx = \begin{vmatrix} \cos x & = & t \\ -\sin x\,dx & = & dt \end{vmatrix}$$

$$= -\int (1-t^2)t^4\,dt = \int (t^6-t^4)\,dt = \frac{t^7}{7} - \frac{t^5}{5} + C = \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$$

2.

$$\int \sin^5 x\,dx = \int (1-\cos^2 x)^2 \sin x\,dx = \begin{vmatrix} \cos x & = & t \\ -\sin x dx & = & dt \end{vmatrix} = -\int (1-t^2)^2\,dt$$

$$= -\int (1-2t^2+t^4)\,dt = -t + \frac{2}{3}t^3 - \frac{1}{5}t^5 + C = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$$

3.

$$\int \sin^5 x \cos^3 x\,dx = \begin{vmatrix} \sin x & = & t \\ \cos x\,dx & = & dt \end{vmatrix} = \int t^5(1-t^2)\,dt = \int (t^5-t^7)\,dt$$

$$= \frac{1}{6}t^6 - \frac{1}{8}t^8 + C = \frac{1}{6}\sin^6 x - \frac{1}{8}\sin^8 x + C$$

Lze použít i substituci $t = \cos x$.

4.

$$\int \frac{\cos^3 x}{\sin^5 x}\,dx = \int \frac{\cos^2 x}{\sin^5 x}\cos x\,dx = \int \frac{1-\sin^2 x}{\sin^5 x}\cos x\,dx = \begin{vmatrix} \sin x & = & t \\ \cos x\,dx & = & dt \end{vmatrix}$$

$$= \int \frac{1-t^2}{t^5}\,dt = \int \left(\frac{1}{t^5} - \frac{1}{t^3}\right) dx = -\frac{1}{4t^4} + \frac{1}{2t^2} + C = -\frac{1}{4\sin^4 x} + \frac{1}{2\sin^2 x} + C$$

5.

$$\int \sin^2 x\,dx = \left| \sin^2 x = \frac{1}{2}(1-\cos 2x) \right| = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

6.

$$\int \cos^2 x\,dx = \left|\cos^2 x = \frac{1}{2}(1+\cos 2x)\right| = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

7.

$$\int \sin^4 x \cos^2 x\,dx = \int \left[\frac{1}{2}(1-\cos 2x)\right]^2 \cdot \frac{1}{2}(1+\cos 2x)\,dx = \left|\begin{array}{lcl} 2x & = & t \\ 2\,dx & = & dt \end{array}\right|$$

$$= \frac{1}{16}\int (1-\cos t-\cos^2 t+\cos^3 t)\,dt = \frac{1}{16}\int (\sin^2 t-\cos t+\cos^3 t)\,dt$$

$$= \frac{1}{16}\left(\int \frac{1}{2}(1-\cos 2t)\,dt - \int \cos t\,dt + \int (1-\sin^2 t)\cos t\,dt\right)$$

$$= \frac{1}{16}\left(\frac{1}{2}t - \frac{1}{4}\sin 2t - \sin t + \sin t - \frac{1}{3}\sin^3 t\right) + C = -\frac{1}{48}\sin^3 2x - \frac{1}{64}\sin 4x + \frac{1}{16}x + C$$

8.

$$\int \frac{1}{\sin x \cos x}\,dx = \int \frac{1}{\operatorname{tg} x \cos^2 x}\,dx = \left|\begin{array}{lcl} \operatorname{tg} x & = & t \\ \frac{1}{\cos^2 x}\,dx & = & dt \end{array}\right| = \int \frac{1}{t}\,dt = \ln|\operatorname{tg} x| + C$$

9.

$$\int \frac{1}{\sin x}\,dx = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx = \frac{1}{2}\int \frac{1}{\operatorname{tg}\frac{x}{2}\cos^2\frac{x}{2}}\,dx = \left|\begin{array}{lcl} \operatorname{tg}\frac{x}{2} & = & t \\ \frac{1}{\cos^2\frac{x}{2}}\frac{1}{2}\,dx & = & dt \end{array}\right|$$

$$= \int \frac{1}{t}\,dt = \ln|\operatorname{tg}\frac{x}{2}| + C$$

10.

$$\int \frac{1}{\cos x}\,dx = \left|\cos x = \sin(x+\frac{\pi}{2})\right| = \int \frac{1}{\sin(x+\frac{\pi}{2})}\,dx = \left|\begin{array}{lcl} x+\frac{\pi}{2} & = & t \\ dx & = & dt \end{array}\right| = \int \frac{1}{\sin t}\,dt$$

$$= \ln\left|\operatorname{tg}\left(\frac{1}{2}(x+\frac{\pi}{2})\right)\right| + C$$

Obecná goniometrická substituce $\operatorname{tg}\frac{x}{2} = t$

$$\operatorname{tg}\frac{x}{2} = t, \quad dx = \frac{2dt}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

1.

$$\int \frac{1-\sin x}{1+\cos x}\,dx = \int \frac{1-\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,dt = \int \frac{1+t^2-2t}{1+t^2+1-t^2}\frac{2}{1+t^2}\,dt = \int \frac{1+t^2-2t}{1+t^2}\,dt$$

$$= \int \left(1-\frac{2t}{1+t^2}\right)dt = t - \ln(1+t^2) + C = \operatorname{tg}\frac{x}{2} - \ln\left(1+\operatorname{tg}^2\frac{x}{2}\right) + C$$

2.

$$\int \frac{dx}{3\sin x - 4\cos x} = \int \frac{1}{3\frac{2t}{1+t^2}+4\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,dt = \int \frac{2}{6t+4-4t^2}\,dt = \int \frac{1}{-2t^2+3t+2}\,dt$$

$$= -\frac{1}{2}\int \frac{1}{t^2-\frac{3}{2}t-1}\,dt = -\frac{1}{2}\int \frac{1}{(t-2)(t+\frac{1}{2})}\,dt = -\frac{1}{2}\int \frac{\frac{2}{5}}{t-2} - \frac{\frac{2}{5}}{t+\frac{1}{2}}\,dt$$

$$= -\frac{1}{5}\ln|t-2| + \frac{1}{5}\ln\left|t+\frac{1}{2}\right| + C = -\frac{1}{5}\ln\left|\operatorname{tg}\frac{x}{2}-2\right| + \frac{1}{5}\ln\left|\operatorname{tg}\frac{x}{2}+\frac{1}{2}\right| + C$$