Příklady s použitím vztahu

$$\lim_{x\to a} c = c,$$

kde c je konstanta.

1.
$$\lim_{x\to 0} 4 = 4$$

2.
$$\lim_{x\to -2} 4 = 4$$

3.
$$\lim_{x\to \infty} 4 = 4$$

4.
$$\lim_{x\to 4} x = 4$$

Příklady s použitím vztahu

$$\lim_{x\to a} f(x) = f(a).$$

1.
$$\lim_{x\to \sqrt{2}} 4x = 4 \lim_{x\to \sqrt{2}} x = 4\cdot\sqrt{2} = 4\sqrt{2}$$

2.
$$\lim_{x\to -\infty} 4x = 4 \lim_{x\to -\infty} x = 4\cdot(-\infty) = -\infty$$

3.
$$\lim_{x\to 4} x^2 = 4^2 = 16$$

4.
$$\lim_{x\to 2}(x^2-1) = 2^2 - 1 = 3$$

5.
$$\lim_{x\to 2} \frac{2x-3}{x^2+1} = \frac{2\cdot 2-3}{2^2+1} = \frac{1}{5}$$

6.
$$\lim_{x\to \frac{\Pi}{3}} \sin x = \frac{\sqrt{3}}{2}$$

Příklady řešené úpravami.

1.
$$\lim_{x\to 0} \frac{x+2x^2}{x} = \lim_{x\to 0} \frac{x(1+2x)}{x} = \lim_{x\to 0}(1+2x) = 1+2\cdot 0 = 1$$

2.
$$\lim_{x\to -1} \frac{x^2+3x+2}{x^3+1} = \lim_{x\to -1} \frac{(x+1)(x+2)}{(x+1)(x^2-x+1)} = \lim_{x\to -1} \frac{x+2}{x^2-x+1} = \frac{-1+2}{(-1)^2-(-1)+1} = \frac{1}{3}$$

3.
$$\lim_{x\to -3} \frac{x^2-9}{x+3} = \lim_{x\to -3} \frac{(x+3)(x-3)}{x+3} = \lim_{x\to -3}(x-3) = -3-3 = -6$$

4.

$$\lim_{x\to 1}\frac{x^3-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{x-1}=\lim_{x\to 1}(x^2+x+1)=1^2+1+1=3$$

5.

$$\lim_{x\to 3}\frac{x^2-4x+3}{x-3}=\lim_{x\to 3}\frac{(x-3)(x-1)}{x-3}=\lim_{x\to 3}(x-1)=3-1=2$$

6.

$$\lim_{x\to -2}\frac{x^2+5x+6}{x^2+x-2}=\lim_{x\to -2}\frac{(x+2)(x+3)}{(x+2)(x-1)}=\lim_{x\to -2}\frac{x+3}{x-1}=\frac{1}{-3}=-\frac{1}{3}$$

7.

$$\lim_{x\to 2}\frac{x^3-5x^2+8x-4}{x^3-3x^2+4}$$

Protože $(x^3-5x^2+8x-4):(x-2)=x^2-3x+2$
a $(x^3-3x^2+4):(x-2)=x^2-x-2$, máme

$$\lim_{x\to 2}\frac{(x-2)(x^2-3x+2)}{(x-2)(x^2-x-2)}=\lim_{x\to 2}\frac{x^2-3x+2}{x^2-x-2}=\lim_{x\to 2}\frac{(x-2)(x-1)}{(x-2)(x+1)}=\lim_{x\to 2}\frac{x-1}{x+1}=\frac{1}{3}$$

Limity řešené pomocí vztahu

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

1.

$$\lim_{x\to 0}\frac{\sin 3x}{5x}=\lim_{x\to 0}\frac{1}{5}\cdot\frac{3\sin 3x}{3x}=\frac{3}{5}\lim_{x\to 0}\frac{\sin 3x}{3x}=\frac{3}{5}\cdot 1=\frac{3}{5}$$

2.

$$\lim_{x\to 0}\frac{2\cos^2 x\sin x}{x}=\lim_{x\to 0}2\cdot\lim_{x\to 0}\cos^2 x\cdot\lim_{x\to 0}\frac{\sin x}{x}=2\cdot\cos^2 0\cdot 1=2\cdot 1\cdot 1=2$$

3.

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x}=\lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}=\lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\frac{1}{1+\cos x}$$

$$=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2\cdot\lim_{x\to 0}\frac{1}{1+\cos x}=1^2\cdot\frac{1}{2}=\frac{1}{2}$$

4.

$$\lim_{x\to 0}\frac{1-\cos 2x}{x\sin x}=\lim_{x\to 0}\frac{1-\cos^2 x+\sin^2 x}{x\sin x}=\lim_{x\to 0}\frac{2\sin^2 x}{x\sin x}=\lim_{x\to 0}\frac{2\sin x}{x}=2\cdot 1=2$$

5.

$$\lim_{x\to 0}\left(\frac{3}{x^2+1}-\frac{x}{\sin x}\right)=\lim_{x\to 0}\frac{3}{x^2+1}-\lim_{x\to 0}\frac{x}{\sin x}=\frac{3}{0^2+1}-\lim_{x\to 0}\frac{1}{\frac{\sin x}{x}}=3-\frac{1}{1}=2$$

6.

$$\lim_{x\to 0}\frac{2\tan^2 x}{3x^2}=\lim_{x\to 0}\frac{2\sin^2 x}{3x^2\cos^2 x}=\lim_{x\to 0}\frac{2}{3\cos^2 x}\cdot\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2=\frac{2}{3\cos^2 0}\cdot 1^2=\frac{2}{3\cdot 1}=\frac{2}{3}$$

7.

$$\lim_{x\to 0}\frac{\sin 3x}{\tan x}=\lim_{x\to 0}\frac{3\sin 3x}{3x}\cdot\frac{x}{\tan x}=3\cdot\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\lim_{x\to 0}\frac{1}{\frac{\tan x}{x}}=3\cdot 1\cdot\lim_{x\to 0}\cos x\cdot\frac{1}{\frac{\sin x}{x}}=3\cdot 1=3$$

Limity s odmocninami řešené úpravou - rozšiřováním vhodným výrazem.

1.

$$\lim_{x\to -1}\frac{x+1}{\sqrt{x+2}-1}=\lim_{x\to -1}\left(\frac{x+1}{\sqrt{x+2}-1}\cdot\frac{\sqrt{x+2}+1}{\sqrt{x+2}+1}\right)=\lim_{x\to -1}\frac{(x+1)(\sqrt{x+2}+1)}{x+2-1}$$

$$=\lim_{x\to -1}\frac{(x+1)(\sqrt{x+2}+1)}{x+1}=\lim_{x\to -1}(\sqrt{x+2}+1)=\sqrt{-1+2}+1=2$$

2.

$$\lim_{x\to 7}\frac{2-\sqrt{x-3}}{x^2-49}=\lim_{x\to 7}\left(\frac{2-\sqrt{x-3}}{x^2-49}\cdot\frac{2+\sqrt{x-3}}{2+\sqrt{x-3}}\right)=\lim_{x\to 7}\frac{4-(x-3)}{(x^2-49)(2+\sqrt{x-3})}$$

$$=\lim_{x\to 7}\frac{7-x}{(x^2-49)(2+\sqrt{x-3})}=\lim_{x\to 7}\frac{-(x-7)}{(x-7)(x+7)(2+\sqrt{x-3})}$$

$$=\lim_{x\to 7}\frac{(-1)}{(x+7)(2+\sqrt{x-3})}=\frac{-1}{14\cdot(2+\sqrt{7-3})}=\frac{-1}{14\cdot 4}=-\frac{1}{56}$$

3.

$$\lim_{x\to 0}\frac{\sin 2x}{\sqrt{x+2}-\sqrt{2}}=\lim_{x\to 0}\left(\frac{\sin 2x}{\sqrt{x+2}-\sqrt{2}}\cdot\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}\right)$$

$$=\lim_{x\to 0}\frac{\sin 2x\cdot(\sqrt{x+2}+\sqrt{2})}{x+2-2}=\lim_{x\to 0}\frac{2\cdot\sin 2x\cdot(\sqrt{x+2}+\sqrt{2})}{2x}$$

$$=2\cdot\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\lim_{x\to 0}(\sqrt{x+2}+\sqrt{2})=2\cdot 1\cdot(\sqrt{2}+\sqrt{2})=4\sqrt{2}$$

Limity řešené pomocí vztahu

$$\lim_{x\to\pm\infty}\frac{1}{x}=0.$$

1.

$$\lim_{x\to+\infty}(5x^3-x^2+2x+1)=\lim_{x\to+\infty}5x^3(1-\frac{x^2}{5x^3}+\frac{2x}{5x^3}+\frac{1}{5x^3})=\lim_{x\to+\infty}5x^3\cdot\lim_{x\to+\infty}(1-\frac{1}{5x}+\frac{2}{5x^2}+\frac{1}{5x^3})$$

$$=\lim_{x\to+\infty}5x^3\cdot 1=\lim_{x\to+\infty}5x^3=+\infty$$

2.

$$\lim_{x\to-\infty}(5x^3-x^2+2x+1)=\lim_{x\to-\infty}5x^3(1-\frac{x^2}{5x^3}+\frac{2x}{5x^3}+\frac{1}{5x^3})=\lim_{x\to-\infty}5x^3\cdot\lim_{x\to-\infty}(1-\frac{1}{5x}+\frac{2}{5x^2}+\frac{1}{5x^3})$$

$$=\lim_{x\to-\infty}5x^3\cdot 1=\lim_{x\to-\infty}5x^3=-\infty$$

3.

$$\lim_{x\to+\infty}(-5x^3-x^2+2x+1)=\lim_{x\to+\infty}-5x^3\cdot\lim_{x\to+\infty}(1+\frac{1}{5x}-\frac{2}{5x^2}-\frac{1}{5x^3})=\lim_{x\to+\infty}-5x^3\cdot 1=-\infty$$

4.

$$\lim_{x\to-\infty}(-5x^3-x^2+2x+1)=\lim_{x\to-\infty}-5x^3\cdot\lim_{x\to-\infty}(1+\frac{1}{5x}-\frac{2}{5x^2}-\frac{1}{5x^3})=\lim_{x\to-\infty}-5x^3\cdot 1=+\infty$$

5.

$$\lim_{x\to+\infty}(2x^4-3x^3+x+1) = \lim_{x\to+\infty} 2x^4(1-\frac{3}{2x}+\frac{1}{2x^3}+\frac{1}{2x^4}) = \lim_{x\to+\infty} 2x^4 = +\infty$$

6.

$$\lim_{x\to-\infty}(2x^4-3x^3+x+1) = \lim_{x\to-\infty} 2x^4(1-\frac{3}{2x}+\frac{1}{2x^3}+\frac{1}{2x^4}) = \lim_{x\to-\infty} 2x^4 = +\infty$$

7.

$$\lim_{x\to+\infty}(-2x^4-3x^3+x+1) = \lim_{x\to+\infty} -2x^4(1+\frac{3}{2x}-\frac{1}{2x^3}-\frac{1}{2x^4}) = \lim_{x\to+\infty} -2x^4 = -\infty$$

8.

$$\lim_{x\to-\infty}(-2x^4-3x^3+x+1) = \lim_{x\to-\infty} -2x^4(1+\frac{3}{2x}-\frac{1}{2x^3}-\frac{1}{2x^4}) = \lim_{x\to-\infty} -2x^4 = -\infty$$

9.

$$\lim_{x\to+\infty} \frac{3x-1}{x^2+1} = \lim_{x\to+\infty} \frac{x^2\cdot(\frac{3}{x}-\frac{1}{x^2})}{x^2\cdot(1+\frac{1}{x})} = \lim_{x\to+\infty} \frac{\frac{3}{x}-\frac{1}{x^2}}{1+\frac{1}{x}} = \frac{0}{1} = 0$$

10.

$$\lim_{x\to+\infty} \frac{5x^2-4x+2}{3x^2+x-3} = \lim_{x\to+\infty} \frac{x^2(5-\frac{4}{x}+\frac{2}{x^2})}{x^2(3+\frac{1}{x}-\frac{3}{x^2})} = \lim_{x\to+\infty} \frac{5-\frac{4}{x}+\frac{2}{x^2}}{3+\frac{1}{x}-\frac{3}{x^2}} = \frac{5}{3}$$

11.

$$\lim_{x\to-\infty} \frac{x^4-5x}{x^2-3x+1} = \lim_{x\to-\infty} \frac{x^2(x^2-\frac{5}{x})}{x^2(1-\frac{3}{x}+\frac{1}{x^2})} = \lim_{x\to-\infty} \frac{x^2-\frac{5}{x}}{1-\frac{3}{x}+\frac{1}{x^2}} = \lim_{x\to-\infty} x^2 = +\infty$$

Vytýkání z pod odmocniny

1.

$$\lim_{x\to+\infty} \frac{\sqrt{x^2-1}+\sqrt{x^2+1}}{x+3} = \lim_{x\to+\infty} \frac{x\left(\sqrt{1-\frac{1}{x^2}}+\sqrt{1+\frac{1}{x^2}}\right)}{x(1+\frac{3}{x})}$$

$$= \lim_{x\to+\infty} \frac{\sqrt{1-\frac{1}{x^2}}+\sqrt{1+\frac{1}{x^2}}}{1+\frac{3}{x}} = \frac{\sqrt{1}+\sqrt{1}}{1} = 2$$

2.

$$\lim_{x\to+\infty} x(\sqrt{x^2+1}-x) = \lim_{x\to+\infty} x(\sqrt{x^2+1}-x)\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} = \lim_{x\to+\infty} \frac{x\cdot(x^2+1-x^2)}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to+\infty} \frac{x}{x\cdot\sqrt{1+\frac{1}{x^2}}+1} = \lim_{x\to+\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$$

3.

$$\lim_{x\to0} \frac{\tan x}{1-\sqrt{1+\tan x}} \quad [\tan x = z, x\to0 \Leftrightarrow z\to0] = \lim_{z\to0} \frac{z}{1-\sqrt{1+z}}$$

$$= \lim_{z\to0} \left(\frac{z}{1-\sqrt{1+z}}\cdot\frac{1+\sqrt{1+z}}{1+\sqrt{1+z}}\right) = \lim_{z\to0} \frac{z(1+\sqrt{1+z})}{1-(1+z)} = -\lim_{z\to0}(1+\sqrt{1+z}) = -(1+\sqrt{1+0}) = -2$$

Limity vycházející z chování exponenciálních funkcí.

1.

$$\lim_{x\to-\infty} 2^{-x} = \lim_{x\to-\infty} \left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^{-\infty} = 2^{\infty} = \infty$$

2.

$$\lim_{x\to+\infty} 3^{-x} = \frac{1}{3^{\infty}} = 0$$

3.

$$\lim_{x\to+\infty} \frac{2^x - 1}{2^x + 1} = \lim_{x\to+\infty} \frac{2^x\left(1 - \frac{1}{2^x}\right)}{2^x\left(1 + \frac{1}{2^x}\right)} = \lim_{x\to+\infty} \frac{1 - \frac{1}{2^x}}{1 + \frac{1}{2^x}} = \frac{1-0}{1+0} = 1$$

4.

$$\lim_{x\to+\infty} \frac{1 - 3^x}{2 + 3^{x+1}} = \lim_{x\to+\infty} \frac{3^{x+1}\left(\frac{1}{3^{x+1}} - \frac{1}{3}\right)}{3^{x+1}\left(\frac{2}{3^{x+1}} + 1\right)} = \frac{0 - \frac{1}{3}}{0+1} = -\frac{1}{3}$$

Limity s využitím vztahů

$$\lim_{x\to+\infty} \left(1 + \frac{1}{x}\right)^x = e$$

a

$$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e.$$

1.

$$\lim_{x\to+\infty} \left(1 + \frac{a}{x}\right)^x = \lim_{x\to+\infty} \left(1 + \frac{1}{\frac{a}{x}}\right)^x = \lim_{x\to+\infty} \left(1 + \frac{1}{\frac{a}{x}}\right)^{a\cdot\frac{x}{a}} = \left(\lim_{x\to+\infty} \left(1 + \frac{1}{\frac{a}{x}}\right)^{\frac{x}{a}}\right)^a = e^a$$

2.

$$\lim_{x\to+\infty} \left(\frac{x}{1+x}\right)^x = \lim_{x\to+\infty} \left(1 - \frac{1}{x+1}\right)^x = \lim_{x\to+\infty} \left(1 - \frac{1}{x+1}\right)^{x+1-1}$$

$$= \lim_{x\to+\infty} \left(1 - \frac{1}{x+1}\right)^{x+1} \cdot \lim_{x\to+\infty} \left(1 - \frac{1}{x+1}\right)^{-1} = e^{-1} \cdot (1-0)^{-1} = \frac{1}{e}$$

3.

$$\lim_{x\to+\infty} \left(\frac{x}{x+5}\right)^{2x-1} = \lim_{x\to+\infty} \left(1 - \frac{5}{x+5}\right)^{2x-1} = \lim_{x\to+\infty} \left(1 - \frac{5}{x+5}\right)^{2(x+5)-11} =$$

$$\left(\lim_{x\to+\infty} \left(1 - \frac{5}{x+5}\right)^{x+5}\right)^2 \cdot \lim_{x\to+\infty} \left(1 - \frac{5}{x+5}\right)^{-11} = (e^{-5})^2 \cdot (1-0)^{-11} = \frac{1}{e^{10}}$$

4.

$$\lim_{x\to+\infty} \left(\frac{2x+4}{2x+5}\right)^{x+3} = \lim_{x\to+\infty} \left(1 - \frac{1}{2x+5}\right)^{x+3} = \lim_{x\to+\infty} \left(1 - \frac{1}{2x+5}\right)^{\frac{2x+6}{2}}$$

$$= \lim_{x\to+\infty} \left(1 - \frac{1}{2x+5}\right)^{\frac{2x+5}{2}} \cdot \lim_{x\to+\infty} \left(1 - \frac{1}{2x+5}\right)^{\frac{1}{2}} = (e^{-1})^{\frac{1}{2}} \cdot 1^{\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

5.

$$\lim_{x\to 0} (1+3x)^{\frac{1}{x}} = \lim_{x\to 0} (1+3x)^{\frac{3}{3x}} = \left(\lim_{x\to 0} (1+3x)^{\frac{1}{3x}}\right)^3 = e^3$$

Existence limity a jednostranné limity.

1.

$$\lim_{x\to 1}\frac{x^2+2x+1}{x^2-4x+3}=\lim_{x\to 1}\frac{x^2+2x+1}{(x-3)(x-1)}=\lim_{x\to 1}\frac{x^2+2x+1}{x-3}\cdot\lim_{x\to 1}\frac{1}{x-1}=-\frac{4}{2}\cdot\lim_{x\to 1}\frac{1}{x-1}=?$$

$$\lim_{x\to 1+}\frac{x^2+2x+1}{x^2-4x+3}=-2\cdot\lim_{x\to 1+}\frac{1}{x-1}=-2\cdot(+\infty)=-\infty$$

$$\lim_{x\to 1-}\frac{x^2+2x+1}{x^2-4x+3}=-2\cdot\lim_{x\to 1-}\frac{1}{x-1}=-2\cdot(-\infty)=+\infty$$

Limita neexistuje.

2.

$$\lim_{x\to 2}\frac{x^2+2x+1}{x^2-4x+4}=\lim_{x\to 2}\frac{x^2+2x+1}{(x-2)^2}=\lim_{x\to 2}(x^2+2x+1)\cdot\lim_{x\to 2}\frac{1}{(x-2)^2}=9\cdot\lim_{x\to 2}\frac{1}{(x-2)^2}=?$$

$$\lim_{x\to 2+}\frac{1}{(x-2)^2}=\infty$$

$$\lim_{x\to 2-}\frac{1}{(x-2)^2}=\infty$$

$$\lim_{x\to 2}\frac{x^2+2x+1}{x^2-4x+4}=9\cdot\infty=\infty$$

3.

$$\lim_{x\to 0}\frac{x}{\sin^2 x}=\lim_{x\to 0}\frac{x}{\sin x}\cdot\lim_{x\to 0}\frac{1}{\sin x}=1\cdot\lim_{x\to 0}\frac{1}{\sin x}$$

$$\lim_{x\to 0+}\frac{1}{\sin x}=\infty$$

$$\lim_{x\to 0-}\frac{1}{\sin x}=-\infty$$

Limita neexistuje.

4.

$$\lim_{x\to 0+}\frac{x}{|x|}=\lim_{x\to 0+}\frac{x}{x}=1$$

$$\lim_{x\to 0-}\frac{x}{|x|}=\lim_{x\to 0+}\frac{x}{-x}=-1$$

5.

$$\lim_{x\to 0+}2^{\frac{1}{x}}=2^{+\infty}=\infty$$

$$\lim_{x\to 0-}2^{\frac{1}{x}}=2^{-\infty}=0$$

6.

$$\lim_{x\to 0+}\frac{2^{\frac{1}{x}}+3}{3^{\frac{1}{x}}+2}=\lim_{x\to 0+}\frac{3^{\frac{1}{x}}\left(\frac{2^{\frac{1}{x}}}{3^{\frac{1}{x}}}+\frac{3}{3^{\frac{1}{x}}}\right)}{3^{\frac{1}{x}}\left(1+\frac{2}{3^{\frac{1}{x}}}\right)}=\lim_{x\to 0+}\frac{\frac{2^{\frac{1}{x}}}{3^{\frac{1}{x}}}+\frac{3}{3^{\frac{1}{x}}}}{1+\frac{2}{3^{\frac{1}{x}}}}=\lim_{x\to 0+}\frac{\left(\frac{2}{3}\right)^{\frac{1}{x}}+\frac{3}{3^{\frac{1}{x}}}}{1+\frac{2}{3^{\frac{1}{x}}}}=\frac{0+0}{1+0}=0$$

$$\lim_{x\to 0-}\frac{2^{\frac{1}{x}}+3}{3^{\frac{1}{x}}+2}=\frac{2^{-\infty}+3}{3^{-\infty}+2}=\frac{0+3}{0+2}=\frac{3}{2}$$